Preprint typeset in JHEP style. - HYPER VERSION

hep-th/0402004
CPHT-RR117.1203
NSF-KITP-04-05
LPTHE-03-38
ROM2F/2003/37

String amplitudes in the Hpp-wave limit of $AdS_3 \times S^3$

M. Bianchi [1], **G. D'Appollonio** [2], **E. Kiritsis** [3,4], **O. Zapata** [1]

[1] *Dipartimento di Fisica, Università di Roma "Tor Vergata"*
I.N.F.N. - Sezione di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1 - 00133 Roma, ITALY

[2] *LPTHE, Université Paris VI, 4 pl Jussieu,*
75252 Paris cedex 05, FRANCE

[3] *CPHT, UMR du CNRS 7644, Ecole Polytechnique,*
91128, Palaiseau, FRANCE

[4] *Department of Physics, University of Crete*
71003 Heraklion, GREECE

E-mails: massimo.bianchi@roma2.infn.it, giuseppe@lpthe.jussieu.fr,
kiritsis@cpht.polytechnique.fr, zapata@roma2.infn.it

Abstract: We compute string amplitudes on pp-waves supported by NS-NS 3-form fluxes and arising in the Penrose limit of $AdS_3 \times S^3 \times \mathcal{M}$. We clarify the role of the non-chiral accidental $SU(2)$ symmetry of the background. We comment on the extension of our results to the superstring and propose a holographic formula in the BMN limit of the AdS_3/CFT_2 correspondence valid for any correlator.

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1. Introduction

The AdS/CFT correspondence suggests a holographic duality between (super)string theory on anti-de Sitter spaces (AdS) and (super)conformal field theories defined on their boundaries. In the past few years, Maldacena's original conjecture has passed

a large number of tests, especially in the AdS_5/CFT_4 case, and has been a precious source of insights on field theories in their (super)conformal phase, along RG flows where (super) conformal symmetry is broken at tree level either spontaneously or explicitly, and to some extent even in cases with a dynamically generated scale (see [2, 3] and reference therein).

The main obstacle towards extending the holographic duality beyond the (super)gravity approximation that captures the strong coupling regime of the boundary (conformal) field theory is represented by our limited understanding of how to quantize the superstring in the presence of R-R backgrounds [4]. One possible exception is the background $AdS_3 \times S^3 \times \mathcal{M}$ supported by a NS-NS 3-form flux which is the near horizon geometry of a bound-state of fundamental strings (F1) and penta-branes (NS5) [5]. Powerful CFT techniques can be exploited in this case to compute the spectrum and string amplitudes since the dynamics on the world-sheet is governed by an $SL(2,\mathbb{R}) \times SU(2)$ WZNW model [8, 9, 10]. S-duality relates NS-NS 3-form flux to R-R flux or a combination of the two and one may in principle resort to the hybrid formalism of Berkovits, Vafa and Witten to make part of the space-time supersymmetries manifest [15] and compute some three-point amplitudes [16].

The dual two-dimensional superconformal field theory is expected to be the non-linear σ-model with target space the symmetric orbifold $\mathcal{M}^N/\mathcal{S}_N$ [17, 2, 5], where $\mathcal{M}$ is taken to be either T^4 or $K3$. Quantitative comparison with boundary conformal field theory predictions is hampered by the presence of non-compact directions in the target space of the non-linear σ-model [10, 11] which would be lifted by turning on a R-R background, *i.e.* moving away from the symmetric orbifold point in the moduli space.

An interesting limit of $AdS \times S$ is the plane wave (pp-wave), which corresponds to the local background seen by an observer moving at the speed of light in the original space. This procedure of zooming-in around a null geodesic is known as Penrose limit [18]. Remarkably many pp-waves are amenable to quantization in the light-cone gauge even in the presence of R-R backgrounds [19]. The string spectrum for $p^+ \neq 0$ can be computed exactly and contrasted with the spectrum of operators with large R-charge J that survive the so-called BMN limit (large N and J with J^2/N fixed). Unfortunately, string interactions and the spectrum at $p^+ = 0$ are hard to determine in the light-cone gauge [22].

Once again it is fruitful to consider the Hpp-wave resulting from the Penrose limit of $AdS_3 \times S^3$ supported by a NS-NS 3-form flux. The world-sheet CFT for the bosonic coordinates is a six-dimensional generalization [20] of the Nappi-Witten (NW) model [23]. The relevant Heisenberg current algebra is $\widehat{\mathcal{H}}_6$, or actually two copies of $\widehat{\mathcal{H}}_6$ with a common central element and a non-chiral external $SU(2)$ automorphism preserved by the limiting flux. This is broken to $U(1)$ when the NS-NS fluxes through the two planes are different, $H_{+12} = \mu_1 \neq H_{+34} = \mu_2$. The theory depends on μ_1/μ_2 only and is exactly solvable for all values of μ_1, μ_2. From the current algebra point of view,

the Penrose limit is carried out by contracting the currents of $\widehat{SL}(2,\mathbb{R})_{k_1} \times \widehat{SU}(2)_{k_2}$ with $\mu_1^2 k_1 = \mu_2^2 k_2$. This marginal deformation interpolates between the generic 6-d Hpp-wave ($\mu_1 \neq \mu_2$), the (super)symmetric one ($\mu_1 = \mu_2$) and the NW model ($\mu_1 = 0$ or $\mu_2 = 0$) or even flat space-time ($\mu_1 = 0$ and $\mu_2 = 0$) very much as the 'null deformation' discussed in [57] interpolates between $AdS_3 \times S^3$ and $R^+ \times S^3$ with a linear dilaton before any Penrose limit is taken.

Exploiting current algebra techniques and a quasi-free field resolution [24], it was possible [1] to explicitly compute string amplitudes in the closely related NW model, that represents the Penrose limit of the near-horizon geometry of a stack of NS5-branes [60, 40] and realizes the $\widehat{\mathcal{H}}_4$ current algebra.

In the present paper, we will apply the same techniques to the pp-wave representing the Penrose limit of $AdS_3 \times S^3 \times \mathcal{M}$. Although we will almost exclusively concentrate our attention on the bosonic string, we will briefly comment on how to extend our results to the superstring. We will compute two-, three- and four-point amplitudes with insertions of tachyon vertex operators of any of the three types of representations of the $\widehat{\mathcal{H}}_6$ current algebra: actually depending on the value of the light-cone momentum p^+, the states belong to discrete representations when $p^+ \neq 0$ or to continuous representations when $p^+ = 0$.

The main novelties we found with respect to the $\widehat{\mathcal{H}}_4$ case are the existence of non-chiral symmetries which correspond to background isometries not realized by the zero-modes of the currents and the presence in the spectrum of new representations of the current algebra that satisfy a modified highest weight condition. The results are compactly encoded in terms of auxiliary charge variables, which form doublets of the external $SU(2)$ symmetry. As expected, the amplitudes computed here by exploiting the $\widehat{\mathcal{H}}_6$ current algebra, coincide with the ones resulting from the Penrose limit, *i.e.* the contraction, of the amplitudes on $AdS_3 \times S^3 \times \mathcal{M}$. This allows us to identify the crucial role played by the charge variables in the fate of holography. They become coordinates on a four-dimensional holographic screen for the pp-wave [40]. Global Ward identities represent powerful constraints on the form of the correlation functions and we would like to argue that higher dimensional generalizations, even in the presence of R-R fluxes where no chiral splitting is expected to take place, should follow the same pattern. We thus believe that some of the pathologies of the BMN limit pointed out in the literature should rather be ascribed to an incomplete knowledge of the scaling limit in the computation of the relevant amplitudes. Taking fully into account the rearrangement, technincally speaking a 'Saletan contraction' [26], of the (super)conformal generators in a $\widehat{\mathcal{H}}_{2+2n}$ Heisenberg algebra is imperative in this sense.

The plan of the present paper is as follows:

In section 2 we briefly describe the Hpp-waves whose σ-models are WZNW models based on the $\mathbf{H}_{2+2n}$ Heisenberg groups and then we concentrate on the six-

dimensional wave that emerges from the Penrose limit of $AdS_3 \times S^3$ discussing the corresponding contraction of the $\widehat{SL}(2,\mathbb{R})_{k_1} \times \widehat{SU}(2)_{k_2}$ currents. In Section 3 we identify the relevant representations of $\widehat{\mathcal{H}}_6^L \times \widehat{\mathcal{H}}_6^R$ and write down the explicit expressions for the tachyon vertex operators. In section 4 we compute two and three-point correlation functions on the world-sheet and compare the results with those obtained from the limit of $AdS_3 \times S^3$. In section 5 we compute four-point correlation functions on the world-sheet by means of current algebra techniques. In section 6 we present the Wakimoto free-field approach and check consistency of the results obtained in this way with those in the previous sections. In section 7 we study string amplitudes in the Hpp-wave and analyze the structure of their singularities. In section 8 we propose a concrete holographic formula relating the Hpp-wave S-matrix elements to precise limits of arbitrary boundary CFT correlators. Finally we draw our conclusions and indicate lines for future investigation.

2. Hpp-waves and the Penrose limit of $AdS_3 \times S^3$

The plane wave backgrounds we will discuss in this paper have the simple form [18]

$$ds^2 = -2dudv - \frac{1}{4}du^2 \sum_{\alpha=1}^{n} \mu_\alpha^2 y_\alpha \bar{y}_\alpha + \sum_{\alpha=1}^{n} dy_\alpha d\bar{y}_\alpha + \sum_{i=1}^{24-2n} g_{ij} dx^i dx^j \tag{2.1}$$

Here u and v are light-cone coordinates, $y_\alpha = r_\alpha e^{i\varphi_\alpha}$ are complex coordinates parameterizing the n transverse planes and x^i are the remaining $24-2n$ dimensions of the critical bosonic string that we assume compactified on some internal manifold $\mathcal{M}$ with metric g_{ij}. In the following we will concentrate on the $2+2n$ dimensional part of the metric in Eq. (2.1). The wave is supported by a non-trivial NS-NS antisymmetric tensor field strength (whence the name Hpp-wave)

$$H = \sum_{\alpha=1}^{n} \mu_\alpha du \wedge dy_\alpha \wedge d\bar{y}_\alpha \ , \tag{2.2}$$

while the dilaton is constant and all the other fields are set to zero.

The background defined in (2.1) and (2.2) with generic μ_α has a $(5n+2)$-dimensional isometry group generated by translations in u and v, independent rotations in each of the n transverse planes and $4n$ "magnetic translations". When $2 \leq k \leq n$ of the μ_α coincide the isometry group is enhanced: the generic $U(1)^n$ rotational symmetry of the metric is enlarged to $SO(2k) \times U(1)^{n-k}$, broken to $U(k) \times U(1)^{n-k}$ by the field strength of the antisymmetric tensor. The dimension of the resulting isometry group is therefore $5n + 2 + k(k-1)$.

As first realized in [23] for the case $n = 1$ and then in [20] for generic n, the σ-models corresponding to Hpp-waves are WZNW models based on the $\mathbf{H}_{2+2n}$ Heisen-

berg group. The $\widehat{\mathcal{H}}_{2+2n}$ current algebra is defined by the following OPEs

$$\begin{aligned} P^+_\alpha(z)P^{-\beta}(w) &\sim \frac{2\delta^\beta_\alpha}{(z-w)^2} - \frac{2i\mu_\alpha\delta^\beta_\alpha}{(z-w)}K(w) \ , \\ J(z)P^+_\alpha(w) &\sim -\frac{i\mu_\alpha}{(z-w)}P^+_\alpha(w) \ , \\ J(z)P^{-\alpha}(w) &\sim +\frac{i\mu_\alpha}{(z-w)}P^{-\alpha}(w) \ , \\ J(z)K(w) &\sim \frac{1}{(z-w)^2} \ , \end{aligned} \tag{2.3}$$

where $\alpha, \beta = 1, ..., n$. The anti-holomorphic currents satisfy a similar set of OPEs [1] and the total affine symmetry of the model is $\widehat{\mathcal{H}}^L_{2+2n} \times \widehat{\mathcal{H}}^R_{2+2n}$.

A few clarifications are in order. First of all the zero modes of the left and right currents only realize a $(4n+3)$-dimensional subgroup of the whole isometry group. The left and right central elements [2] K and $\bar{K}$ are identified and generate translations in v; P^+_α and $P^{-\alpha}$ together with their right counterparts generate the $4n$ magnetic translations; $J+\bar{J}$ generates translations in u and $J-\bar{J}$ a simultaneous rotation in all the n transverse planes. In the following we will refer to the subgroup of the isometry group that is not generated by the zero modes of the currents as G_I.

The position of the index $\alpha = 1, ..., n$ carried by the $P^\pm$ generators is meant to emphasize that at the point where the generic $U(1)^n$ part of the isometry group is enhanced to $U(n) = SU(n)_I \times U(1)_{J-\bar{J}}$ they transform respectively in the fundamental and in the anti-fundamental representation of $SU(n)_I$. The left and right current modes satisfy the same commutation relations with the generators of the $SU(n)_I$ symmetry of the background.

Let us discuss some particular cases. When $n = 1$ we have the original NW model and all the background isometries are realized by the zero-modes of the currents. When $n = 2$ there is an additional $U(1)_I$ symmetry which extends to $SU(2)_I$ for $\mu_1 = \mu_2$. In this paper we will describe in detail only the six-dimensional Hpp-wave, because of its relation to the BMN limit of the AdS_3/CFT_2 correspondence. Higher-dimensional Hpp-waves do not display any new special feature. When we discuss the Wakimoto representation for the $\mathbf{H}_6$ WZNW model, the following change of variables

$$y^\alpha = e^{i\mu_\alpha u/2}w^\alpha \ , \qquad \bar{y}_\alpha = e^{-i\mu_\alpha u/2}\bar{w}_\alpha \ , \tag{2.4}$$

which yields a metric with a $U(2)$ invariant form

$$ds^2 = -2dudv + \frac{i}{4}du\sum_{\alpha=1}^{2}\mu_\alpha(w^\alpha d\bar{w}_\alpha - \bar{w}_\alpha dw^\alpha) + \sum_{i=1}^{2} dw^\alpha d\bar{w}_\alpha \ . \tag{2.5}$$

[1] As usual we will distinguish the right objets by putting a bar on them.

[2] Notice that we use the same letter for both a (spin s) current $W(z)$ and the corresponding charge $W \equiv W_0 = \oint \frac{dz}{2\pi i} z^{s-1}W(z)$. In order to avoid any confusion we try always to emphasize the two-dimensional nature of the world-sheet fields by showing their explicit z dependence.

will prove useful.

As it is well known, the background (2.1), (2.2) with $n = 2$ and $\mu_1 = \mu_2$ arises from the Penrose limit of $AdS_3 \times S^3$, the near horizon geometry of an $F1|NS5$ bound state. The general metric with $\mu_1 \neq \mu_2$ can also be obtained as a Penrose limit but starting with different curvatures for AdS_3 and S^3. In global coordinates the metric can be written as

$$ds_6^2 = R_1^2\,[-(\cosh\rho)^2 dt^2 + d\rho^2 + (\sinh\rho)^2 d\varphi_1^2] + R_2^2\,[(\cos\theta)^2 d\psi^2 + d\theta^2 + (\sin\theta)^2 d\varphi_2^2]\ , \tag{2.6}$$

where (t, ρ, φ_1) parameterize the three dimensional anti-de Sitter space with curvature radius R_1 and $(\theta, \psi, \varphi_2)$ parameterize S^3 with curvature radius R_2. In the Penrose limit we focus on a null geodesic of a particle moving along the axis of AdS_3 ($\rho \to 0$) and spinning around the equator of the three sphere ($\theta \to 0$). We then change variables according to

$$t = \frac{\mu_1 u}{2} + \frac{v}{\mu_1 R_1^2} \qquad \psi = \frac{\mu_2 u}{2} - \frac{v}{\mu_2 R^2} \qquad \rho = \frac{r_1}{R_1} \qquad \theta = \frac{r_2}{R_2}\ , \tag{2.7}$$

and take the limit sending $R_1, R_2 \to \infty$ while keeping $\mu_1^2 R_1^2 = \mu_2^2 R_2^2$.

From the world-sheet point of view, the Penrose limit of $AdS_3 \times S^3$ amounts to a contraction of the current algebra of the underlying $\widehat{SL}(2,\mathbb{R}) \times \widehat{SU}(2)$ WZNW model. The $\widehat{SL}(2,\mathbb{R})$ current algebra at level k_1 is given by

$$\begin{aligned}
K^+(z)K^-(w) &\sim \frac{k_1}{(z-w)^2} - \frac{2K^3(w)}{z-w}\ , \\
K^3(z)K^\pm(w) &\sim \pm\frac{K^\pm(w)}{z-w}\ , \\
K^3(z)K^3(w) &\sim -\frac{k_1}{2(z-w)^2}\ .
\end{aligned} \tag{2.8}$$

Similarly the $\widehat{SU}(2)$ current algebra at level k_2 is

$$\begin{aligned}
J^+(z)J^-(w) &\sim \frac{k_2}{(z-w)^2} + \frac{2J^3(w)}{z-w}\ , \\
J^3(z)J^\pm(w) &\sim \pm\frac{J^\pm(w)}{z-w}\ , \\
J^3(z)J^3(w) &\sim \frac{k_2}{2(z-w)^2}\ .
\end{aligned} \tag{2.9}$$

The contraction to the $\widehat{\mathcal{H}}_6$ algebra defined in (2.3) is performed by first introducing the new currents

$$\begin{aligned}
P_1^\pm &= \sqrt{\frac{2}{k_1}}K^\pm\ , \qquad\qquad P_2^\pm = \sqrt{\frac{2}{k_2}}J^\pm\ , \\
J &= -i(\mu_1 K^3 + \mu_2 J^3)\ , \qquad K = -i\left(\frac{K^3}{\mu_1 k_1} - \frac{J^3}{\mu_2 k_2}\right)\ ,
\end{aligned} \tag{2.10}$$

and then by taking the limit $k_1, k_2 \to \infty$ with $\mu_1^2 k_1 = \mu_2^2 k_2$.

In view of possible applications of our analysis to the superstring, and in order to be able to consider flat space or a torus with $c_{int} = 20$ as a consistent choice for the internal manifold $\mathcal{M}$ of the bosonic string before the Penrose limit is taken, one should choose $k_1 - 2 = k_2 + 2 = k$ so that the central charge is $c = 6$.

3. Spectrum of the model

Our aim in this section is to determine the spectrum of the string in the Hpp-wave with $\mathbf{H}_6$ Heisenberg symmetry. As in the $\mathbf{H}_4$ case, in addition to 'standard' highest-weight representations, new modified highest-weight (MHW) representations should be included. In the $\mathbf{H}_4$ case as well as in $\mathbf{H}_6$ with $SU(2)_I$ symmetry, such MHW representations are actually spectral flowed representations. However, in the general $\mathbf{H}_6$ $\mu_1 \neq \mu_2$ case, we have the novel phenomenon that spectral flow cannot generate the MHW representations.

The MHW representations are difficult to handle in the current algebra formalism. Fortunately they are easy to analyze in the quasi-free field representation [24, 1] where their unitarity and their interactions are straightforward.

3.1 $\mathcal{H}_6$ representations

The representation theory of the extended Heisenberg algebras, such as $\mathcal{H}_6$, is very similar to the $\mathcal{H}_4$ case [24, 1]. The $\mathcal{H}_6$ commutation relations are

$$[P_\alpha^+, P^{-\beta}] = -2i\mu_\alpha \delta_\alpha^\beta K \ , \quad [J, P_\alpha^+] = -i\mu_\alpha P_\alpha^+ \ , \quad [J, P^{-\alpha}] = i\mu_a P^{-\alpha} \ . \tag{3.1.1}$$

As explained in the previous paragraph this algebra generically admits an additional $U(1)_I$ generator I^3 that satisfies

$$[I^3, P_\alpha^+] = -i(\sigma^3)_\alpha{}^\beta P_\beta^+ \ , \qquad [I^3, P^{-\alpha}] = i(\sigma^{3,t})^\alpha{}_\beta P^{-\beta} \ . \tag{3.1.2}$$

When $\mu_1 = \mu_2 \equiv \mu$ the $U(1)_I$ symmetry is enhanced to $SU(2)_I$

$$[I^a, P_\alpha^+] = -i(\sigma^a)_\alpha{}^\beta P_\beta^+ \ , \qquad [I^a, P^{-\alpha}] = i(\sigma^{a,t})^\alpha{}_\beta P^{-\beta} \ , \quad a = 1, 2, 3 \ . \tag{3.1.3}$$

For $\mathcal{H}_6$ there are two Casimir operators: the central element K and the combination

$$\mathcal{C} = 2JK + \frac{1}{2}\sum_{\alpha=1}^{2}(P_\alpha^+ P^{-\alpha} + P^{-\alpha} P_\alpha^+) \ . \tag{3.1.4}$$

There are three types of unitary representations:

1) Lowest-weight representations $V_{p,\hat{\jmath}}^+$, where $p > 0$. They are constructed starting from a state $|p, \hat{\jmath}\rangle$ which satisfies $P_\alpha^+|p, \hat{\jmath}\rangle = 0$, $K|p, \hat{\jmath}\rangle = ip|p, \hat{\jmath}\rangle$ and $J|p, \hat{\jmath}\rangle =$

$i\hat{\jmath}|p,\hat{\jmath}\rangle$. The spectrum of J is given by $\{\hat{\jmath}+\mu_1 n_1+\mu_2 n_2\}$, $n_1,n_2\in\mathbb{N}$ and the value of the Casimir is $\mathcal{C}=-2p\hat{\jmath}+(\mu_1+\mu_2)p$.

2) Highest-weight representations $V^-_{p,\hat{\jmath}}$, where $p>0$. They are constructed starting from a state $|p,\hat{\jmath}\rangle$ which satisfies $P^-_\alpha|p,\hat{\jmath}\rangle=0$, $K|p,\hat{\jmath}\rangle=-ip|p,\hat{\jmath}\rangle$ and $J|p,\hat{\jmath}\rangle=i\hat{\jmath}|p,\hat{\jmath}\rangle$. The spectrum of J is given by $\{\hat{\jmath}-\mu_1 n_1-\mu_2 n_2\}$, $n_1,n_2\in\mathbb{N}$ and the value of the Casimir is $\mathcal{C}=2p\hat{\jmath}+(\mu_1+\mu_2)p$. The representation $V^-_{p,-\hat{\jmath}}$ is conjugate to $V^+_{p,\hat{\jmath}}$.

3) Continuous representations $V^0_{s_1,s_2,\hat{\jmath}}$ with $p=0$. These representations are characterized by $K|s_1,s_2,\hat{\jmath}\rangle=0$, $J|s_1,s_2,\hat{\jmath}\rangle=i\hat{\jmath}|s_1,s_2,\hat{\jmath}\rangle$ and $P^\pm_\alpha|s_1,s_2,\hat{\jmath}\rangle\neq 0$. The spectrum of J is then given by $\{\hat{\jmath}+\mu_1 n_1+\mu_2 n_2\}$, with $n_1,n_2\in\mathbb{Z}$ and $|\hat{\jmath}|\leq\frac{\mu}{2}$ where $\mu=\min(\mu_1,\mu_2)$. In this case we have two other Casimirs besides K: $\mathcal{C}_1=P^+_1P^{-1}$ and $\mathcal{C}_2=P^+_2P^{-2}$. Their values are $\mathcal{C}_\alpha=s^2_\alpha$, with $s_\alpha\geq 0$ and $\alpha=1,2$. The one dimensional representation can be considered as a particular continuous representation, where the charges s_α and $\hat{\jmath}$ are zero.

The ground states of all these representations are assumed to be invariant under the $U(1)_I$ ($SU(2)_I$) symmetry. This follows from comparison with the spectrum of the scalar Laplacian in the gravitational wave background, described below.

Since we are dealing with infinite dimensional representations, it is very convenient to introduce charge variables in order to keep track of the various components of a given representation in a compact form. We introduce two doublets of charge variables x_α and x^α, $\alpha=1,2$. The action of the $\mathcal{H}_6$ generators and of the additional generator I^3 on the $V^+_{p,\hat{\jmath}}$ representations is given by

$$
\begin{aligned}
P^+_\alpha &= \sqrt{2}\mu_\alpha p x_\alpha\ , \qquad P^{-\alpha}=\sqrt{2}\partial^\alpha\ , \qquad K=ip\ ,\\
J &= i\left(\hat{\jmath}+\mu_\alpha x_\alpha\partial^\alpha\right)\ , \qquad I^3=ix_\alpha(\sigma^{3,t})^\alpha{}_\beta\partial^\beta\ .
\end{aligned}
\tag{3.1.5}
$$

Similarly for the $V^-_{p,\hat{\jmath}}$ representations we have

$$
\begin{aligned}
P^+_\alpha &= \sqrt{2}\partial_\alpha\ , \qquad P^{-\alpha}=\sqrt{2}\mu_\alpha p x^\alpha\ , \qquad K=-ip\ ,\\
J &= i\left(\hat{\jmath}-\mu_\alpha x^\alpha\partial_\alpha\right)\ , \qquad I^3=-ix^\alpha(\sigma^{3,t})^\alpha{}_\beta\partial^\beta\ .
\end{aligned}
\tag{3.1.6}
$$

Finally for the $V^0_{s_1,s_2,\hat{\jmath}}$ representations we have

$$
P^+_\alpha=s_\alpha x_\alpha\ ,\quad P^{-\alpha}=s_\alpha x^\alpha\ ,\quad J=i\left(\hat{\jmath}+\mu_\alpha x_\alpha\partial^\alpha\right)\ ,\quad I^3=ix_\alpha(\sigma^{3,t})^\alpha{}_\beta\partial^\beta\ ,
\tag{3.1.7}
$$

with the constraints $x^1x_1=x^2x_2=1$, *i.e.* $x_\alpha=e^{i\phi_\alpha}$. Alternative representations of the generators are possible. In particular, acting on $V^0_{s,\hat{\jmath}}$, it may prove convenient to introduce charge variables ξ_α such that $\sum_\alpha\xi_\alpha\xi^\alpha=1$. The ξ_α are related to the x_α in (3.1.7) by $\xi_\alpha=\frac{s_\alpha}{s}x_\alpha$ where $s^2=s_1^2+s_2^2$.

We can easily organize the spectrum of the D'Alembertian in the plane wave background in representations of $\mathcal{H}^L_6\times\mathcal{H}^R_6$. Using radial coordinates in the two

transverse planes the covariant scalar D'Alembertian reads

$$\nabla^2 = -2\partial_u\partial_v + \sum_{\alpha=1}^{2}\left(\partial_{r_\alpha}^2 + \frac{1}{r_\alpha^2}\partial_{\varphi_\alpha}^2 + \frac{1}{r_\alpha}\partial_{r_\alpha} + \frac{\mu_\alpha^2}{4}r_\alpha^2\partial_v^2\right) \ , \tag{3.1.8}$$

and its scalar eigenfunctions may be taken to be of the form

$$f_{p^+,p^-}(u,v,r_\alpha,\varphi_\alpha) = e^{ip^+v+ip^-u}g(r_\alpha,\varphi_\alpha) \ . \tag{3.1.9}$$

For $p^+ \neq 0$, $g(r_\alpha,\varphi_\alpha)$ is given by the product of wave-functions for two harmonic oscillators in two dimensions with frequencies $\omega_\alpha = |p^+|\,\mu_\alpha/2$

$$g_{l_\alpha,m_\alpha}(r_\alpha,\varphi_\alpha) = \left(\frac{l_\alpha!}{2\pi(l_\alpha+|m_\alpha|)!}\right)^{\frac{1}{2}} e^{im_\alpha\varphi_\alpha}e^{-\frac{\xi_\alpha}{2}}\xi_\alpha^{\frac{|m_\alpha|}{2}}L_{l_\alpha}^{|m_\alpha|}(\xi_\alpha) \ , \tag{3.1.10}$$

with $\xi_\alpha = \frac{\mu_\alpha p^+ r_\alpha^2}{2}$ and $l_\alpha \in \mathbb{N}$, $m_\alpha \in \mathbb{Z}$. The resulting eigenvalue is

$$\Lambda_{p^+\neq 0} = 2p^+p^- - \sum_{\alpha=1}^{2}\mu_\alpha\left|p^+\right|(2l_\alpha+|m_\alpha|+1) \ . \tag{3.1.11}$$

and by comparison with the value of the Casimir on the $\mathcal{H}_6^L \times \mathcal{H}_6^R$ representations we can identify

$$p = \left|p^+\right| \ , \qquad \hat{\jmath} = p^- - \sum_{\alpha=1}^{2}\mu_\alpha(2l_\alpha+|m_\alpha|) \ , \qquad m_\alpha = n_\alpha - \bar{n}_\alpha \ , \qquad l_\alpha = \mathrm{Max}(n_\alpha,\bar{n}_\alpha) \ . \tag{3.1.12}$$

For $p^+ = 0$ the $g(r_\alpha,\varphi_\alpha)$ can be taken to be Bessel functions and they give the decomposition of a plane wave whose radial momentum in the two transverse planes is s_α^2, $\alpha = 1,2$.

3.2 $\widehat{\mathcal{H}}_6$ representations and long strings

The representations of the affine Heisenberg algebra $\widehat{\mathcal{H}}_6$ that will be relevant for the study of string theory in the six-dimensional Hpp-wave are the highest-weight representations with a unitary base and some new representations with a modified highest-weight condition that we will introduce below and that in the case $\mu_1 = \mu_2$ coincide with the spectral flowed representations.

The OPEs in (2.3) correspond to the following commutation relations for the $\widehat{\mathcal{H}}_6^L$ left-moving current modes

$$[P_{\alpha n}^+, P_m^{-\beta}] = 2n\delta_\alpha^\beta\,\delta_{n+m} - 2i\mu_\alpha\delta_\alpha^\beta K_{n+m} \ , \qquad [J_n, K_m] = n\delta_{n+m,0} \ ,$$

$$[J_n, P_{\alpha m}^+] = -i\mu_\alpha P_{\alpha\,n+m}^+ \ , \qquad [J_n, P_m^{-\alpha}] = i\mu_\alpha P_{n+m}^{-\alpha} \ . \tag{3.2.1}$$

There are three types of highest-weight representations. Affine representations based on $V^{\pm}_{p,\hat{\jmath}}$ representations of the horizontal algebra, with conformal dimension

$$h = \mp p\hat{\jmath} + \frac{\mu_1 p}{2}(1-\mu_1 p) + \frac{\mu_2 p}{2}(1-\mu_2 p) \ , \tag{3.2.2}$$

and affine representations based on $V^0_{s_1,s_2,\hat{\jmath}}$ representations, with conformal dimension

$$h = \frac{s_1^2}{2} + \frac{s_2^2}{2} = \frac{s^2}{2} \ . \tag{3.2.3}$$

In the current algebra formalism we can introduce a doublet of charge variables and regroup the infinite number of fields that appear in a given representation of $\widehat{\mathcal{H}}^L_6$ in a single field

$$\Phi^+_{p,\hat{\jmath}}(z;x_\alpha) = \sum_{n_1,n_2=0}^{\infty} \prod_{\alpha=1}^{2} \frac{(x_\alpha\sqrt{\mu_\alpha p})^{n_\alpha}}{\sqrt{n_\alpha!}} R^+_{p,\hat{\jmath};n_1,n_2}(z) \ , \quad p>0 \ , \tag{3.2.4}$$

$$\Phi^-_{p,\hat{\jmath}}(z;x^\alpha) = \sum_{n_1,n_2=0}^{\infty} \prod_{\alpha=1}^{2} \frac{(x^\alpha\sqrt{\mu_\alpha p})^{n_\alpha}}{\sqrt{n_\alpha!}} R^-_{p,\hat{\jmath};n_1,n_2}(z) \ , \quad p>0 \ , \tag{3.2.5}$$

$$\Phi^0_{s_1,s_2,\hat{\jmath}}(z;x_\alpha) = \sum_{n_1,n_2=-\infty}^{\infty} \prod_{\alpha=1}^{2} (x_\alpha)^{n_\alpha} R^0_{s_1,s_2,\hat{\jmath};n_1,n_2}(z) \ , \quad s_1,s_2 \geq 0 \ . \tag{3.2.6}$$

Highest-weight representations of the current algebra lead to a string spectrum free from negative norm states only if they satisfy the constraint

$$\mathrm{Max}(\mu_1 p, \mu_2 p) < 1 \ . \tag{3.2.7}$$

When $\mu_1 = \mu_2 = \mu$ new representations should be considered that result from spectral flow of the original representations [8]. Spectral flowed representations are highest-weight representations of an isomorphic algebra whose modes are related to the original ones by

$$\begin{aligned} \tilde{P}^+_{\alpha,n} &= P^+_{\alpha,n-w} \ , \qquad \tilde{P}^{-\alpha}_n = P^{-\alpha}_{n+w} \ , \qquad \tilde{J}_n = J_n \ , \\ \tilde{K}_n &= K_n - iw\delta_{n,0} \ , \qquad \tilde{L}_n = L_n - iwJ_n \ . \end{aligned} \tag{3.2.8}$$

The long strings in this case can move freely in the two transverse planes and correspond to the spectral flowed type 0 representations, exactly as for the $\mathbf{H}_4$ NW model [1].

In the general case $\mu_1 \neq \mu_2$ a similar interpretation is not possible. However instead of introducing new representations as spectral flowed representations we can still define them through a modified highest-weight condition. Such Modified Highest Weight (MHW) representations are a more general concept compared to spectral flowed representations, as the analysis for $\mu_1 \neq \mu_2$ indicates.

In order to understand which kind of representations are needed for the description of states with p outside the range (3.2.7), it is useful to resort to a free field realization of the $\widehat{\mathcal{H}}_{2+2n}$ algebras, first introduced for the original NW model in [24]. This representation provides an interesting relation between primary vertex operators and twist fields in orbifold models. For $\widehat{\mathcal{H}}_6$ we introduce a pair of free bosons $u(z), v(z)$ with $\langle v(z)u(w)\rangle = \log(z-w)$ and two complex bosons $y_\alpha(z) = \xi_\alpha(z) + i\eta_\alpha(z)$ and $\tilde{y}^\alpha(z) = \xi_\alpha(z) - i\eta_\alpha(z)$ with $\langle y_\alpha(z)\tilde{y}^\beta(w)\rangle = -2\delta_\alpha^\beta \log(z-w)$. The currents

$$\begin{aligned} J(z) &= \partial v(z) \ , & K(z) &= \partial u(z) \ , \\ P^+_\alpha(z) &= ie^{-i\mu_\alpha u(z)}\partial y_\alpha(z) \ , & P^{-\alpha}(z) &= ie^{i\mu_\alpha u(z)}\partial \tilde{y}^\alpha(z) \ , \end{aligned} \tag{3.2.9}$$

satisfy the $\widehat{\mathcal{H}}_6$ OPEs (2.3). The ground state of a $V^\pm_{p,\hat{\jmath}}$ representation is given by the primary field

$$R^\pm_{p,\hat{\jmath};0}(z) = e^{i[\hat{\jmath}u(z) \pm pv(z)]}\sigma^\mp_{\mu_1 p}(z)\sigma^\mp_{\mu_2 p}(z) \ . \tag{3.2.10}$$

The $\sigma^\mp_{\mu p}(z)$ are twist fields, characterized by the following OPEs

$$\begin{aligned} \partial y(z)\sigma^-_{\mu p}(w) &\sim (z-w)^{-\mu p}\ \tau^-_{\mu p}(w) \ , & \partial \tilde{y}(z)\sigma^-_{\mu p}(w) &\sim (z-w)^{-1+\mu p}\ \sigma^{-(1)}_{\mu p}(w) \ , \\ \partial y(z)\sigma^+_{\mu p}(w) &\sim (z-w)^{-1+\mu p}\ \sigma^{+(1)}_{\mu p}(w) \ , & \partial \tilde{y}(z)\sigma^+_{\mu p}(w) &\sim (z-w)^{-\mu p}\ \tau^+_{\mu p}(w) \ , \end{aligned} \tag{3.2.11}$$

where $\tau^\pm_{\mu p}(z)$ and $\sigma^{\pm(1)}_{\mu p}(z)$ are excited twist fields. The ground state of a $V^0_{s_1,s_2,\hat{\jmath}}$ representation is determined by the primary field

$$R^0_{s_1,s_2,\hat{\jmath};0}(z) = e^{i\hat{\jmath}u(z)}R^0_{s_1}(z)R^0_{s_2}(z) \ , \tag{3.2.12}$$

where

$$R^0_{s_\alpha}(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta_\alpha e^{\frac{is_\alpha}{2}\left(y_\alpha(z)e^{-i\theta_\alpha} + \tilde{y}^\alpha(z)e^{i\theta_\alpha}\right)} \ . \tag{3.2.13}$$

are essentially free vertex operators.

In analogy with $\widehat{\mathcal{H}}_4$ we define for arbitrary $\mu p > 0$

$$\begin{aligned} R^\pm_{p,\hat{\jmath};0}(z) &= e^{i[\hat{\jmath}u(z)\pm pv(z)]}\sigma^\mp_{\{\mu_1 p\}}(z)\sigma^\mp_{\{\mu_2 p\}}(z) \ , & \{\mu_1 p\} \neq 0 \ , \{\mu_2 p\} \neq 0 \ , \\ R^\pm_{p,\hat{\jmath},s_1;0}(z) &= e^{i[\hat{\jmath}u(z)\pm pv(z)]}R^0_{s_1}(z)\sigma^\mp_{\{\mu_2 p\}}(z) \ , & \{\mu_1 p\} = 0 \ , \{\mu_2 p\} \neq 0 \ , \\ R^\pm_{p,\hat{\jmath},s_2;0}(z) &= e^{i[\hat{\jmath}u(z)\pm pv(z)]}\sigma^\mp_{\{\mu_1 p\}}(z)R^0_{s_2}(z) \ , & \{\mu_1 p\} \neq 0 \ , \{\mu_2 p\} = 0 \ , \end{aligned} \tag{3.2.14}$$

where $[\mu p]$ and $\{\mu p\}$ are the integer and fractional part of μp respectively. Quantization of the model in the light-cone gauge shows that the resulting string spectrum is unitary. From the current algebra point of view the states that do not satisfy the bound (3.2.7) belong to new representations which satisfy a modified highest-weight condition and are defined as follows. When $K_0|p,\hat{\jmath}\rangle = i\mu p|p,\hat{\jmath}\rangle$ with $\{\mu_\alpha p\} \neq 0$, $\alpha = 1,2$, the affine representations we are interested in are defined by

$$\begin{aligned} P^+_{\alpha,\ n}|p,\hat{\jmath}\rangle &= 0 \ , \quad n \geq -[\mu_\alpha p] \ , & P^{-\alpha}_n|p,\hat{\jmath}\rangle &= 0 \ , \quad n \geq 1 + [\mu_\alpha p] \ , \\ J_n|p,\hat{\jmath}\rangle &= 0 \ , \quad n \geq 1 \ , & K_n|p,\hat{\jmath}\rangle &= 0 \ , \quad n \geq 1 \ . \end{aligned} \tag{3.2.15}$$

Similarly when $K_0|p,\hat{\jmath}\rangle = -i\mu p|p,\hat{\jmath}\rangle$ with $\{\mu_\alpha p\} \neq 0$, $\alpha = 1, 2$, the affine representations we are interested in are defined by

$$\begin{aligned} P^+_{\alpha,\,n}|p,\hat{\jmath}\rangle = 0 \;, \quad n \geq 1 + [\mu_\alpha p] \;, \qquad & P^{-\alpha}_n|p,\hat{\jmath}\rangle = 0 \;, \quad n \geq -[\mu_\alpha p] \;, \\ J_n|p,\hat{\jmath}\rangle = 0 \;, \quad n \geq 1 \;, \qquad & K_n|p,\hat{\jmath}\rangle = 0 \;, \quad n \geq 1 \;. \end{aligned} \tag{3.2.16}$$

Finally whenever either $\{\mu_1 p\} = 0$ or $\{\mu_2 p\} = 0$ we introduce new ground states $|p, s_1, \hat{\jmath}\rangle$ and $|p, s_2, \hat{\jmath}\rangle$ which satisfy the same conditions as in (3.2.15), (3.2.16) except that

$$P^+_{\alpha,\,n}|p,\hat{\jmath},s_\alpha\rangle = 0 \;, \quad n \geq -[\mu_\alpha p] \;, \qquad P^{-\alpha}_n|p,\hat{\jmath},s_\alpha\rangle = 0 \;, \quad n \geq [\mu_\alpha p] \;, \tag{3.2.17}$$

for either $\alpha = 1$ or $\alpha = 2$.

These states correspond to strings that do not feel any more the confining potential in one of the two transverse planes. The presence of these states in the spectrum can be justified along similar lines as for AdS_3 [8] or the $\mathbf{H}_4$ [40, 1] WZNW models.

4. Three-point functions

We now turn to compute the simplest interactions in the Hpp-wave, encoded in the three-point functions of the scalar (tachyon) vertex operators identified in the previous section. We will initially discuss the non symmetric $\mu_1 \neq \mu_2$ case, where global Ward identities can be used to completely fix the form of the correlators. We will then address the $SU(2)_I$ symmetric case and argue that the requirement of non-chiral $SU(2)_I$ invariance is crucial in getting a unique result. We will finally describe the derivation of the two and three-point functions starting from the corresponding quantities in $AdS_3 \times S^3$.

4.1 $\mathcal{H}_6$ three-point couplings

In the last section we have seen that the primary fields of the $\widehat{\mathcal{H}}_6^L \times \widehat{\mathcal{H}}_6^R$ affine algebra are of the form

$$\Phi^a_\nu(z,\bar{z};x,\bar{x}) \;, \tag{4.1.1}$$

where $a = \pm, 0$ labels the type of representation and ν stands for the charges that are necessary in order to completely specify the representation, *i.e.* $\nu = (p,\hat{\jmath})$ for $V^\pm$ and $\nu = (s_1, s_2, \hat{\jmath})$ for V^0. Finally x stands for the charge variables we introduced to keep track of the states that form a given representation: $x = x_\alpha$ for V^+, $x = x^\alpha$ for V^- and $x = x_\alpha$ with $x_\alpha = 1/x^\alpha$ (*i.e.* $x_\alpha = e^{i\phi_\alpha}$) for V^0 . In the following we will leave the dependence of the vertex operators on the anti-holomorphic variables $\bar{z}$ and $\bar{x}$ understood. The OPE between the currents and the primary vertex operators can be written in a compact form

$$\mathcal{J}^A(z)\Phi^a_\nu(w;x) = \mathcal{D}^A_a \frac{\Phi^a_\nu(w;x)}{z-w} \;, \tag{4.1.2}$$

where A labels the six $\widehat{\mathcal{H}}_6$ currents and the $\mathcal{D}^A_a$ are the differential operators that realize the action of $\mathcal{J}^A_0$ on a given representation (a,ν), according to (3.1.5), (3.1.6) and (3.1.7).

We fix the normalization of the operators in the $V^{\pm}_{p_1,\hat{\jmath}_1}$ representations by choosing the overall constants in their two-point functions, which are not determined by the world-sheet or target space symmetries, to be such that

$$\langle \Phi^+_{p_1,\hat{\jmath}_1}(z_1,x_{1\alpha})\Phi^-_{p_2,\hat{\jmath}_2}(z_2,x_2^\alpha)\rangle = \frac{|\prod_{\alpha=1}^2 e^{-p_1\mu_\alpha x_{1\alpha}x_2^\alpha}|^2}{|z_{12}|^{4h}}\delta(p_1-p_2)\delta(\hat{\jmath}_1+\hat{\jmath}_2)\ , \qquad (4.1.3)$$

where we introduced the shorthand notation $f(z,x)f(\bar{z},\bar{x}) = |f(z,x)|^2$. Similarly, the other non-trivial two-point functions are chosen to be

$$\langle \Phi^0_{s_{1\alpha},\hat{\jmath}_1}(z_1,x_{1\alpha})\Phi^0_{s_{2\alpha},\hat{\jmath}_2}(z_2,x_{2\alpha})\rangle = \prod_{\alpha=1,2}\frac{\delta(s_{1\alpha}-s_{2\alpha})}{s_{1\alpha}}\delta(\phi_{1\alpha}-\phi_{2\alpha})\delta(\bar{\phi}_{1\alpha}-\bar{\phi}_{2\alpha})\delta(\hat{\jmath}_1+\hat{\jmath}_2)\ , \qquad (4.1.4)$$

where we set $x_{i\alpha} = e^{i\phi_{i\alpha}}$.

Three-point functions, denoted by $G_{abc}(z_i,x_i)$ or more simply by $\langle abc\rangle$ in the following, are determined by conformal invariance on the world-sheet to be of the form

$$\langle \Phi^a_{\nu_1}(z_1,x_1)\Phi^b_{\nu_2}(z_2,x_2)\Phi^c_{\nu_3}(z_3,x_3)\rangle = \frac{C_{abc}(\nu_1,\nu_2,\nu_3)K_{abc}(x_1,x_2,x_3)}{|z_{12}|^{2(h_1+h_2-h_3)}|z_{13}|^{2(h_2+h_3-h_2)}|z_{23}|^{2(h_2+h_3-h_1)}}\ , \qquad (4.1.5)$$

where C_{abc} are the quantum structure constants of the CFT and the 'kinematical' coefficients K_{abc} contain all the dependence on the $\mathcal{H}^L_6\times\mathcal{H}^R_6$ charge variables x and $\bar{x}$. For generic values of μ_1 and μ_2 ($\frac{\mu_1}{\mu_2}\notin\mathbb{Q}$), the functions K_{abc} are completely fixed by the global Ward identities, as it was the case for the $\mathbf{H}_4$ WZNW model [1]. When $\mu_1=\mu_2$ we will have to impose the additional requirement of $SU(2)_I$ invariance. An important piece of information for understanding the structure of the three-point couplings is provided by the decomposition of the tensor products between representations of the $\mathcal{H}_6$ horizontal algebra

$$\begin{aligned}
V^+_{p_1,\hat{\jmath}_1}\otimes V^+_{p_2,\hat{\jmath}_2} &= \sum_{n_1,n_2=0}^{\infty} V^+_{p_1+p_2,\hat{\jmath}_1+\hat{\jmath}_2+\mu_1 n_1+\mu_2 n_2}\ ,\\
V^+_{p_1,\hat{\jmath}_1}\otimes V^-_{p_2,\hat{\jmath}_2} &= \sum_{n_1,n_2=0}^{\infty} V^+_{p_1+p_2,\hat{\jmath}_1+\hat{\jmath}_2-\mu_1 n_1-\mu_2 n_2}\ , \qquad p_1>p_2\ ,\\
V^+_{p_1,\hat{\jmath}_1}\otimes V^-_{p_2,\hat{\jmath}_2} &= \sum_{n_1,n_2=0}^{\infty} V^-_{p_1+p_2,\hat{\jmath}_1+\hat{\jmath}_2+\mu_1 n_1+\mu_2 n_2}\ , \qquad p_1<p_2\ .
\end{aligned} \qquad (4.1.6)$$

Note that when $\mu_1=\mu_2$ there are $n+1$ terms with the same $\hat{\jmath}=\hat{\jmath}_1+\hat{\jmath}_2\pm\mu n$ in (4.1.6). The existence of this multiplicity is precisely what is necessary in order to

obtain $SU(2)_I$ invariant couplings, as we will explain in the following. We will also need

$$\begin{aligned} V^+_{p,\hat{j}_1} \otimes V^-_{p,\hat{j}_2} &= \int_0^\infty s_1 ds_1 \int_0^\infty s_2 ds_2 V^0_{s_1,s_2,\hat{j}_1+\hat{j}_2} \; , \\ V^+_{p_1,\hat{j}_1} \otimes V^0_{s_1,s_2,\hat{j}_2} &= \sum_{n_1,n_2=-\infty}^{\infty} V^+_{p_1+p_2,\hat{j}_1+\hat{j}_2+\mu_1 n_1+\mu_2 n_2} \; . \end{aligned} \tag{4.1.7}$$

Let us first discuss the generic case $\mu_1 \neq \mu_2$, starting from $\langle ++- \rangle$. According to (4.1.6) this coupling is non-vanishing only when $p_1+p_2 = p_3$ and $L = -(\hat{j}_1+\hat{j}_2+\hat{j}_3) = \mu_1 q_1 + \mu_2 q_2$, with $q_1, q_2 \in \mathbb{N}$. The global Ward identities can be unambiguously solved and the result is[3]

$$K_{++-}(q_1, q_2) = \left| \prod_{\alpha=1}^{2} e^{-\mu_\alpha x_3^\alpha (p_1 x_{1\alpha} + p_2 x_{2\alpha})} (x_{2\alpha} - x_{1\alpha})^{q_\alpha} \right|^2 \; . \tag{4.1.8}$$

The corresponding three-point couplings are

$$C_{++-}(q_1, q_2) = \prod_{\alpha=1}^{2} \frac{1}{q_\alpha!} \left[\frac{\gamma(\mu_\alpha p_3)}{\gamma(\mu_\alpha p_1)\gamma(\mu_\alpha p_2)} \right]^{\frac{1}{2}+q_\alpha} \; , \tag{4.1.9}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. All other couplings that only involve $\Phi^\pm$ vertex operators follow from (4.1.8), (4.1.9) by permutation of the indices and by using the fact that $K_{++-}C_{++-} \to K_{--+}C_{--+}$ up to the exchange $x_i^\alpha \leftrightarrow x_{i\alpha}$ and the inversion of the signs of all the $\hat{j}_i$.

Similarly the $\langle + - 0 \rangle$ coupling can be non-zero only when $p_1 = p_2$ and $L = -(\hat{j}_1 + \hat{j}_2 + \hat{j}_3) = \sum_\alpha \mu_\alpha q_\alpha$, with $q_\alpha \in \mathbb{Z}$. Global Ward identities yield

$$K_{+-0} = \left| \prod_{\alpha=1}^{2} e^{-\mu_\alpha p_1 x_{1\alpha} x_2^\alpha - \frac{s_\alpha}{\sqrt{2}} \left(x_2^\alpha x_{3\alpha} + x_{1\alpha} x_3^\alpha \right)} x_{3\alpha}^{q_\alpha} \right|^2 \; . \tag{4.1.10}$$

Moreover

$$C_{+-0}(p, \hat{j}_1; p, \hat{j}_2; s_1, s_2, \hat{j}_3) = \prod_{\alpha=1}^{2} e^{\frac{s_\alpha^2}{2}[\psi(\mu_\alpha p) + \psi(1-\mu_\alpha p) - 2\psi(1)]} \; , \tag{4.1.11}$$

where $\psi(x) = \frac{d \ln \Gamma(x)}{dx}$ is the digamma function.

Finally the coupling between three Φ^0 vertex operators simply reflects momentum conservation in the two transverse planes. Therefore it is non-zero only when

$$s_{3\alpha}^2 = s_{1\alpha}^2 + s_{2\alpha}^2 + 2 s_{1\alpha} s_{2\alpha} \cos\xi_\alpha \; , \qquad s_{3\alpha} e^{i\eta_\alpha} = -s_{1\alpha} - s_{2\alpha} e^{i\xi_\alpha} \; , \qquad \alpha = 1, 2 \; , \tag{4.1.12}$$

[3]The standard δ-function for the Cartan conservation rules are always implied. We do not write them explicitly.

where $\xi_\alpha = \phi_{2\alpha} - \phi_{1\alpha}$ and $\eta_\alpha = \phi_{3\alpha} - \phi_{1\alpha}$. It can be written as

$$K_{000}(\phi_{1\alpha}, \phi_{2\alpha}, \phi_{3\alpha}) = \prod_{\alpha=1}^{2} \frac{8\pi^2 \delta(\xi_\alpha + \bar{\xi}_\alpha)\delta(\eta_\alpha + \bar{\eta}_\alpha)}{\sqrt{4s_{1\alpha}^2 s_{2\alpha}^2 - (s_{3\alpha}^2 - s_{1\alpha}^2 - s_{2\alpha}^2)^2}} e^{-iq_\alpha(\phi_{1\alpha} + \bar{\phi}_{1\alpha})} \ , \quad (4.1.13)$$

where the angles ξ_α and η_α are fixed by the Eqs. (4.1.12) and again $L = \sum_\alpha \mu_\alpha q_\alpha$ with $q_\alpha \in \mathbb{Z}$.

As discussed in section 2, when $\mu_1 = \mu_2 = \mu$ the plane wave background displays an additional $SU(2)_I$ symmetry. At the same time we see from (4.1.6) that there are also new possible couplings and they precisely combine to give an $SU(2)_I$ invariant result. Let us start again from three-point couplings containing only $\Phi^\pm$ vertex operators. In this case the $SU(2)_I$ invariant result is obtained after summing over all the couplings $C_{++-}(q_1, q_2)$ with $(q_1 + q_2) = L/\mu = Q$

$$\begin{aligned} K_{++-}(Q)C_{++-}(Q) &= \sum_{q_1=0}^{Q} K_{++-}(q_1, Q-q_1)C_{++-}(q_1, Q-q_1) \quad (4.1.14) \\ &= \frac{1}{Q!}\left[\frac{\gamma(\mu p_3)}{\gamma(\mu p_1)\gamma(\mu p_2)}\right]^{\frac{1}{2}+Q} \left|e^{-\mu\sum_{\alpha=1}^{2} x_3^\alpha(p_1 x_{1\alpha} + p_2 x_{2\alpha})}\right|^2 ||x_2 - x_1||^{2Q} \ , \end{aligned}$$

where $||x||^2 \equiv \sum_\alpha |x_\alpha|^2$ is indeed $SU(2)_I$ invariant.

Similarly the $\langle + - 0\rangle$ correlator becomes, after summing over $q_1 \in \mathbb{Z}$,

$$\begin{aligned} K_{+-0}(Q)C_{+-0}(p, \hat{\jmath}_1; p, \hat{\jmath}_2; s_1, s_2, \hat{\jmath}_3) &= \prod_{\alpha=1}^{2} \left|e^{-\mu p_1 x_{1\alpha} x_2^\alpha - \frac{s_\alpha}{\sqrt{2}}\left(x_2^\alpha x_{3\alpha} + x_{1\alpha} x_3^\alpha\right)}\right|^2 \left(\frac{||x_3||^2}{2}\right)^Q \\ e^{\frac{s_1^2+s_2^2}{2}[\psi(\mu p) + \psi(1-\mu p) - 2\psi(1)]} \ , \quad (4.1.15) \end{aligned}$$

with the constraint $x_{31}\bar{x}_3^1 = x_{32}\bar{x}_3^2$. The $\langle 000\rangle$ coupling gets similarly modified.

4.2 The Penrose limit of the charge variables

It is interesting to discuss how the three-point couplings in the Hpp-wave with $\mathcal{H}_6^L \times \mathcal{H}_6^R$ symmetry are related to the three-point couplings in $AdS_3 \times S^3$. The first thing we have to understand is how the $\mathcal{H}_6$ representations arise in the limit from representations of $SL(2,\mathbb{R}) \times SU(2)$. For $SU(2)$ we have the representations $V(\tilde{l})$ with $2\tilde{l} \in \mathbb{N}$ and $\tilde{m} = -\tilde{l}, -\tilde{l}+1, ..., \tilde{l}$. For $SL(2,\mathbb{R})$ we have three types of unitary normalizable representations:

1) Lowest-weight discrete representations $\mathcal{D}^+(l)$, constructed starting form a state $|l\rangle$ which satisfies $K^-|l\rangle = 0$, with $l > 1/2$. The spectrum of K^3 is given by $\{l+n\}$, $n \in \mathbb{N}$ and the Casimir is $\mathcal{C}_{SL} = -l(l-1)$.

2) Highest-weight discrete representations $\mathcal{D}^-(l)$, constructed starting form a state $|l\rangle$ which satisfies $K^+|l\rangle = 0$, with $l > 1/2$. The spectrum of K^3 is given by $\{-l-n\}$, $n \in \mathbb{N}$ and the Casimir is $\mathcal{C}_{SL} = -l(l-1)$.

3) Continuous representations $\mathcal{D}^0(l,\alpha)$, constructed starting form a state $|l,\alpha\rangle$ which satisfies $K^{\pm}|l,\alpha\rangle \neq 0$, with $l = 1/2 + i\sigma$, $\sigma \geq 0$. The spectrum of K^3 is given by $\{\alpha + n\}$, $n \in \mathbb{Z}$ and $0 \leq \alpha < 1$. The Casimir is $\mathcal{C}_{SL} = 1/4 + \sigma^2$.

Let us start with the $V^+_{p,\hat{\jmath}}$ representations. Following [1] we consider states that sit near the top of an $SU(2)$ representation

$$\tilde{l} = \frac{k_2}{2}\mu_2 p - b \ , \qquad \tilde{m} = \frac{k_2}{2}\mu_2 p - b - n_2 \ . \tag{4.2.1}$$

In order to get in the limit states with a finite conformal dimension and well defined quantum numbers with respect to the currents in (2.10), we have to choose for $SL(2,\mathbb{R})$ a $\mathcal{D}^-(l)$ representation with

$$l = \frac{k_1}{2}\mu_1 p - a \ , \qquad m = -\frac{k_1}{2}\mu_1 p + a - n_1 \ . \tag{4.2.2}$$

In the limit $\hat{\jmath} = -\mu_1 a + \mu_2 b$. Reasoning in a similar way one can see that the $V^-_{p,\hat{\jmath}}$ representations result from $\mathcal{D}^+(l) \times V(\tilde{l})$ representations with

$$\begin{aligned} l &= \frac{k_1}{2}\mu_1 p - a \ , \qquad & m &= \frac{k_1}{2}\mu_1 p - a + n_1 \ , \\ \tilde{l} &= \frac{k_2}{2}\mu_2 p - b \ , \qquad & \tilde{m} &= -\frac{k_2}{2}\mu_2 p + b + n_2 \ , \end{aligned} \tag{4.2.3}$$

and $\hat{\jmath} = \mu_1 a - \mu_2 b$ in the limit. Finally the $V^0_{s_1,s_2,\hat{\jmath}}$ representations result from $\mathcal{D}^0(l,\alpha) \times V(\tilde{l})$ representations with

$$l = \frac{1}{2} + i\sqrt{\frac{k_1}{2}}s_1 \ , \quad m = \alpha + n_1 \ , \quad \tilde{l} = \sqrt{\frac{k_2}{2}}s_2 \ , \quad \tilde{m} = n_2 \ , \tag{4.2.4}$$

and $\hat{\jmath} = -\mu_1 \alpha$. The tensor product of these representations reproduces in the limit the ones displayed before for $\mathcal{H}_6$ in Eq. (4.1.6).

Let us briefly discuss how the Penrose limit acts on the wave-functions corresponding to the representations considered above. We will consider only the limit of the ground states but the analysis can be easily extended to the limit of the whole $SL(2,\mathbb{R}) \times SU(2)$ representation if we introduce a generating function for the matrix elements, which can be expressed in terms of the Jacobi functions.

Using global coordinates for $AdS_3 \times S^3$, the ground state of a $\mathcal{D}^-_l \times V(\tilde{l})$ representation can be written as

$$e^{2ilt - 2i\tilde{l}\psi}(\cosh\rho)^{-2l}(\cos\theta)^{2\tilde{l}} \ . \tag{4.2.5}$$

After scaling the coordinates and the quantum numbers as required by the Penrose limit this function becomes

$$e^{2ipv + i\hat{\jmath}u - \frac{p}{2}(\mu_1 r_1^2 + \mu_2 r_2^2)} \ , \qquad \hat{\jmath} = -\mu_1 a + \mu_2 b \ . \tag{4.2.6}$$

In the same way starting from a $\mathcal{D}_l^+ \times V(\tilde{l})$ representation

$$e^{-2ilt+2i\tilde{l}\psi}(\cosh\rho)^{-2l}(\cos\theta)^{2\tilde{l}} \ , \tag{4.2.7}$$

we obtain

$$e^{-2ipv+i\hat{\jmath}u-\frac{p}{2}(\mu_1 r_1^2+\mu_2 r_2^2)} \ , \qquad \hat{\jmath} = \mu_1 a - \mu_2 b \ . \tag{4.2.8}$$

As anticipated, the limit of the generating functions lead to semiclassical wave-functions for the six-dimensional wave which are a simple generalizations of those displayed in [1].

We introduce a vertex operator for each unitary representations of $SL(2,\mathbb{R})$

$$\begin{aligned}
\Psi_l^+(z,x) &= \sum_{n=0}^{\infty} c_{l,n}(-x)^n R_{l,n}^+(z) \ , \\
\Psi_l^-(z,x) &= \sum_{n=0}^{\infty} c_{l,n} x^{-2l-n} R_{l,n}^-(z) \ , \\
\Psi_{l,\alpha}^0(z,x) &= \sum_{n\in\mathbb{Z}} x^{-l+\alpha+n} R_{l,\alpha,n}^0(z) \ ,
\end{aligned} \tag{4.2.9}$$

where $c_{l,n}^2 = \frac{\Gamma(2l+n)}{\Gamma(n+1)\Gamma(2l)}$. The differential operators that represent the $SL(2,\mathbb{R})$ action are

$$\mathcal{D}_1^- = -x^2\partial_x - 2lx \ , \quad \mathcal{D}_1^+ = -\partial_x \ , \quad \mathcal{D}_1^3 = l + x\partial_x \ . \tag{4.2.10}$$

Similarly for S^3 we introduce

$$\Omega_{\tilde{l}}(z,y) = \sum_{m=-\tilde{l}}^{\tilde{l}} \tilde{c}_{\tilde{l},m} y^{\tilde{l}+m} R_{\tilde{l},m}(z) \ , \tag{4.2.11}$$

where $\tilde{c}_{\tilde{l},m}^2 = \frac{\Gamma(2\tilde{l}+1)}{\Gamma(\tilde{l}+m+1)\Gamma(\tilde{l}-m+1)}$ and the differential operators that represent the $SU(2)$ action are

$$\mathcal{D}_2^+ = \partial_y \ , \quad \mathcal{D}_2^- = -y^2\partial_y + 2\tilde{l}y \ , \quad \mathcal{D}_2^3 = y\partial_y - \tilde{l} \ . \tag{4.2.12}$$

Generalizing the case studied in [1], we can now implement the Penrose limit on the operators $\Psi_l^a(z,x)\Omega_{\tilde{l}}(z,y)$ and determine their precise relation with the $\hat{\mathcal{H}}_6$ operators $\Phi^a(z,x,y)$. In this section we shall denote the two $\mathcal{H}_6$ charge variables as x and y in order to emphasize that they are related to the charge variables of $SL(2,\mathbb{R})$ and $SU(2)$ respectively. For the discrete representations we have

$$\Phi_{p,\hat{\jmath}}^+(z,x,y) = \lim_{k_1,k_2\to\infty} \left(\frac{x}{\sqrt{k_1}}\right)^{-2l} \left(\frac{y}{\sqrt{k_2}}\right)^{2\tilde{l}} \Psi_l^-\left(z,\frac{\sqrt{k_1}}{x}\right) \Omega_{\tilde{l}}\left(z,\frac{\sqrt{k_2}}{y}\right) \tag{4.2.13}$$

$$\Phi_{p,\hat{\jmath}}^-(z,x,y) = \lim_{k_1,k_2\to\infty} \Psi_l^+\left(z,-\frac{x}{\sqrt{k_1}}\right) \Omega_{\tilde{l}}\left(z,\frac{y}{\sqrt{k_2}}\right) \ , \tag{4.2.14}$$

with

$$l = \frac{k_1}{2}\mu_1 p - a \ , \qquad \tilde{l} = \frac{k_2}{2}\mu_2 p - b \ . \tag{4.2.15}$$

For the continuous representations we have

$$\Phi^0_{s_1,s_2,\hat{\jmath}}(z,x,y) = \lim_{k_1,k_2\to\infty} (-ix)^{-l+\alpha}\, y^{\tilde{l}}\ \Psi^0_{l,\alpha}\left(z,\frac{i}{x}\right)\Omega_{\tilde{l}}\left(z,\frac{1}{y}\right) \ , \tag{4.2.16}$$

with

$$l = \frac{1}{2} + i\sqrt{\frac{k_1}{2}}s_1 \ , \qquad \tilde{l} = \sqrt{\frac{k_2}{2}}s_2 \ . \tag{4.2.17}$$

With the help of the previous formulae it is not difficult to find the Clebsch-Gordan coefficients of the plane-wave three-point correlators. In fact, a similar analysis has been performed in [1] for the three-point correlators of the Nappi-Witten gravitational wave considered as a limit of $SU(2)_k \times U(1)$. For AdS_3 the general form of the three point function is fixed, up to normalization, by $\widehat{SL}(2,\mathbb{R})_L \times \widehat{SL}(2,\mathbb{R})_R$ invariance (x dependence) and by $SL(2,\mathbb{C})$ global conformal invariance on the world-sheet (z dependence), to be

$$\left\langle \prod_{i=1}^{3} \Psi_{l_1}(z_i,\bar{z}_i,x_i,\bar{x}_i) \right\rangle = C(l_1,l_2,l_3) \prod_{i<j}^{1,3} \frac{1}{|x_{ij}|^{2l_{ij}}|z_{ij}|^{2h_{ij}}} \ , \tag{4.2.18}$$

where $l_{12} = l_1 + l_2 - l_3$, $h_{12} = h_1 + h_2 - h_3$ and cyclic permutation of the indexes. Due to the $\widehat{SU}(2)_L \times \widehat{SU}(2)_R$ and world-sheet conformal invariance the correlation function of three primaries on S^3 is given by

$$\left\langle \prod_{i=1}^{3} \Omega_{\tilde{l}_i}(z_i,\bar{z}_i,y_i,\bar{y}_i) \right\rangle = C(\tilde{l}_1,\tilde{l}_2,\tilde{l}_3) \prod_{i<j}^{1,3} \frac{|y_{ij}|^{2\tilde{l}_{ij}}}{|z_{ij}|^{2h_{ij}}} \ , \tag{4.2.19}$$

where $\tilde{l}_{ij}$ and h_{ij} are defined as for (4.2.18). Let us consider for instance the limit leading to a $\langle ++-\rangle$ correlator. Taking into account that $\sum_i \hat{\jmath}_i = -L = -\mu_1(a_1 + a_2 - a_3) + \mu_2(b_1 + b_2 - b_3)$, the kinematic coefficient receives the following contribution from the AdS_3 part

$$K_{++-}(x,\bar{x}) = k_1^{-q_1} \left|e^{-\mu_1 x_3(p_1 x_1 + p_2 x_2)}\right|^2 |x_2 - x_1|^{2q_1} \ , \tag{4.2.20}$$

where $q_1 = a_1 + a_2 - a_3$ and a similar contribution from the S^3 part

$$K_{++-}(y,\bar{y}) = k_2^{-q_2} \left|e^{-\mu_2 y_3(p_1 y_1 + p_2 y_2)}\right|^2 |y_2 - y_1|^{2q_2} \ , \tag{4.2.21}$$

where $q_2 = -b_1 - b_2 + b_3$. Putting the two contributions together

$$K_{++-}(x,\bar{x},y,\bar{y}) = k_1^{-q_1}k_2^{-q_2} \left|e^{-\mu_1 x_3(p_1 x_1 + p_2 x_2)}e^{-\mu_2 y_3(p_1 y_1 + p_2 y_2)}\right|^2 |x_2 - x_1|^{2q_1}|y_2 - y_1|^{2q_2} \ , \tag{4.2.22}$$

we reproduce (4.1.8). In the $SU(2)$ invariant case $\mu_1 = \mu_2$, one finds a looser constraint on the a_i and b_i that leads to $q_1+q_2 = Q = -L/\mu$. Summing over the allowed values of q_1 and q_2 one eventually gets the $SU(2)_I$ invariant result (4.1.14). Using the above expression for the CG coefficients for a coupling of the form $\langle + - 0\rangle$ one obtains

$$K_{+-0}(x,\bar{x},y,\bar{y}) = \left|e^{-\mu_1 p_1 x_1 x_2 - \frac{s_1}{\sqrt{2}}(x_2 x_3 + x_1 x_3)}\right|^2 \left|e^{-\mu_2 p_1 y_1 y_2 - \frac{s_2}{\sqrt{2}}(y_2 y_3 + y_1 y_3)}\right|^2 |x_3|^{2q_1}|y_3|^{2q_2}\ , \tag{4.2.23}$$

where $q_1 = a_1 - a_2 + \alpha$ and $q_2 = b_2 - b_1$.

4.3 The Penrose limit of the $AdS_3 \times S^3$ three-point couplings

We now turn to the Penrose limit of the $AdS_3 \times S^3$ structure constants. The limit of the $SU(2)$ three-point couplings [36] has been considered in [1] and we refer to that paper for a detailed discussion. Here we provide a similar analysis for the $SL(2,\mathbb{R})$ structure constants [37] and show that when combined with the $SU(2)$ part they reproduce in the limit the $\hat{\mathcal{H}}_6$ structure constants.

In general, the AdS_3/CFT_2 correspondence entails the exact equivalence between (super)string theory on $AdS_3 \times \mathcal{K}$, where $\mathcal{K}$ is some compact space represented by a unitary CFT on the worldsheet, and a CFT defined on the boundary of AdS_3. Equivalence at the quantum level implies a isomorphism of the Hilbert spaces and of the operator algebras of the two theories. For various reasons it is often convenient to consider the Euclidean version of AdS_3 described by an $SL(2,\mathbb{C})/SU(2)$ WZNW model on the hyperbolic space H_3^+ with S^2 boundary. Although the Lorentzian $SL(2,\mathbb{R})$ WZNW model and the Euclidean $SL(2,\mathbb{C})/SU(2)$ WZNW model are formally related by analytic continuation of the string coordinates, their spectra are quite distinct. As observed in [8, 10], except for unflowed ($w = 0$) continuous representations, physical string states on Lorentzian AdS_3 corresponds to non-normalizable states in the Euclidean $SL(2,\mathbb{C})/SU(2)$ model. Yet unitarity of the dual boundary CFT_2 that follows from positivity of the Hamiltonian and slow growth of the density of states should make the analytic continuation legitimate. Indeed correlation functions for the Lorentzian $SL(2,\mathbb{R})$ WZNW model have been obtained by analytic continuation of those for the Euclidean $SL(2,\mathbb{C})/SU(2)$ WZNW model [10]. Singularities displayed by correlators involving non-normalizable states have been given a physical interpretation both at the level of the worldsheet, as due to worldsheet instantons, and of the target space. Some singularities have been associated to operator mixing and other to the non-compactness of the target space of the boundary CFT_2. The failure of the factorization of some four-point string amplitudes has been given an explanation in [10] and argued not to prevent the validity of the analytic continuation from Euclidean to Lorentzian signature. Since we are going to take a Penrose limit of $SL(2,\mathbb{R})$ correlation functions computed by analytic continuation from $SL(2,\mathbb{C})/SU(2)$, we need to assume the validity of this procedure. Reversing the

argument, the agreement we found between correlation functions in the Hpp-wave computed by current algebra techniques or by the Wakimoto representation with those resulting from the Penrose limit (current contraction) of the $SL(2,\mathbb{C})/SU(2)$ WZNW model should be taken as further evidence for the validity of the analytic continuation.

For the euclidean AdS_3, that is the H_3^+ WZNW model, the two and three-point functions involving vertex operators in unitary representations were computed by Teschner [37]. The two-point functions are given by

$$\langle \Psi_{l_1}(x_1,z_1)\Psi_{l_2}(x_2,z_2)\rangle = \frac{1}{|z_{12}|^{4h_{l_1}}}\left[\frac{\delta^2(x_1-x_2)\delta(l_1+l_2-1)}{B(l_1)} + \frac{\delta(l_1-l_2)}{|x_{12}|^{4l_1}}\right] , \quad (4.3.1)$$

where

$$B(l) = \frac{\nu^{1-2l}}{\pi b^2\gamma(b^2(2l-1))} , \qquad \nu = \pi\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)} , \qquad b^2 = \frac{1}{k_1-2} , \quad (4.3.2)$$

and $l = \frac{1}{2} + i\sigma$. The three-point functions have the same dependence on the z_i and the x_i as displayed in (4.2.18). The structure constants are given by

$$\begin{aligned} C(l_1,l_2,l_3) = & -\frac{b^2 Y_b(b)}{2\sqrt{\pi\nu}\gamma(1+b^2)}\prod_{i=1}^{3}\frac{\sqrt{\gamma(b^2(2l_i-1))}}{G_b(1-2l_i)}\times \quad (4.3.3)\\ & \times G_b(1-l_1-l_2-l_3)G_b(l_3-l_1-l_2)G_b(l_2-l_1-l_3)G_b(l_1-l_2-l_3) . \end{aligned}$$

In the previous expression we used the entire function $Y_b(z)$ introduced in [66] and the closely related function $G_b(z)$ given by

$$G_b(z) = \frac{b^{-b^2 z\left(z+1+\frac{1}{b^2}\right)}}{Y_b(-bz)} . \quad (4.3.4)$$

The function Y_b satisfies

$$Y_b(z+b) = \gamma(bz)Y_b(z)b^{1-2bz} , \qquad Y_b(z) = Y_b(b+1/b-z) . \quad (4.3.5)$$

In order to study the Penrose limit of the $SL(2,\mathbb{R})$ structure constants we express the function $G_b(z)$ in term of the function $P_b(z)$ that appears in the $SU(2)$ three-point functions [36] and whose asymptotic behaviour was studied in [1]. For this purpose we write

$$\ln P_b(z) = f(b^2,b^2|z) - f(1-zb^2,b^2|z) , \quad (4.3.6)$$

where $f(a,b\,|z)$ is the Dorn-Otto function [65] and then use the relation

$$f(bu,b^2|z) - f(bv,b^2|z) = \ln Y_b(v) - \ln Y_b(u) + zb(u-v)\ln b , \qquad u+v = b+\frac{1}{b}-zb . \quad (4.3.7)$$

The result is

$$G_b(z) = \frac{b\gamma(-b^2 z)}{Y_b(b)P_b(-z)} \ , \tag{4.3.8}$$

and we can rewrite the coupling (4.3.3) using the function P_b

$$\begin{aligned} C(l_1, l_2, l_3) = & -\frac{b^3}{2\sqrt{\pi\nu}\gamma(1+b^2)} \prod_{i=1}^{3} \frac{P_b(2l_i - 1)}{\sqrt{\gamma(b^2(2l_i - 1))}} \times \\ & \times \frac{\gamma(b^2(l_1+l_2+l_3-1))\gamma(b^2(l_1+l_2-l_3))\gamma(b^2(l_1+l_3-l_2))\gamma(b^2(l_2+l_3-l_1))}{P_b(l_1+l_2+l_3-1)P_b(l_1+l_2-l_3)P_b(l_1+l_3-l_2)P_b(l_2+l_3-l_1)} \ . \end{aligned} \tag{4.3.9}$$

Let us consider first the $\langle ++- \rangle$ coupling. As we explained before, the AdS_3 quantum numbers have to be scaled as follows

$$l_i = \frac{k_1}{2}\mu_1 p_i - a_i \ . \tag{4.3.10}$$

The leading behaviour is

$$C(l_1, l_2, l_3) \sim \frac{1}{2\pi b q_1} \frac{1}{P_b(-q_1)} \left[\frac{\gamma(\mu_1 p_3)}{\gamma(\mu_1 p_1)\gamma(\mu_1 p_2)} \right]^{\frac{1}{2}+q_1} \ , \tag{4.3.11}$$

where $q_1 = a_1 + a_2 - a_3$. Due to the presence of $P_b(-q_1)$ in the denominator, the coupling vanishes unless $q_1 \in \mathbb{N}$, thus reproducing the classical tensor products (4.1.6). We can then write

$$\lim_{b\to 0} C(l_1, l_2, l_3) = (-1)^{q_1} \frac{k_1^{q_1+\frac{1}{2}}}{q_1!} \left[\frac{\gamma(\mu_1 p_3)}{\gamma(\mu_1 p_1)\gamma(\mu_1 p_2)} \right]^{\frac{1}{2}+q_1} \sum_{n\in\mathbb{N}} \delta(q_1 - n) \ . \tag{4.3.12}$$

The sign $(-1)^{q_1}$ does not appear in the $\mathcal{H}_6$ couplings, a discrepancy which might be due to some difference between the charge variables used in [37] and the charge variables used in the present paper. The same limit for the $SU(2)$ three-point couplings leads to

$$\lim_{\tilde{b}\to 0} \tilde{C}(\tilde{l}_1, \tilde{l}_2, \tilde{l}_3) = \frac{k_2^{q_2+\frac{1}{2}}}{q_2!} \left[\frac{\gamma(\mu p_3)}{\gamma(\mu p_1)\gamma(\mu p_2)} \right]^{\frac{1}{2}+q_2} \sum_{n\in\mathbb{N}} \delta(q_2 - n) \tag{4.3.13}$$

where $\tilde{b}^{-2} = k_2 + 2$ and $q_2 = -b_1 - b_2 + b_3$. We then reproduce the coupling in 4.1.9. Proceeding in a similar way for a $\langle + - 0 \rangle$ correlator we obtain from AdS_3

$$\lim_{b\to 0} C(l_1, l_2, l_3) = \frac{2^{-is_1\sqrt{2k_1}}}{\sqrt{2\pi}} e^{\frac{s_1^2}{2}(\psi(p)+\psi(1-p)-2\psi(1))} \ , \tag{4.3.14}$$

and similarly from S^3

$$\lim_{\tilde{b}\to 0} \tilde{C}(\tilde{l}_1, \tilde{l}_2, \tilde{l}_3) = \frac{2^{1+s_2\sqrt{2k_2}}}{\sqrt{2\pi}} e^{\frac{s_2^2}{2}(\psi(p)+\psi(1-p)-2\psi(1))} \ . \tag{4.3.15}$$

5. Four-point functions

Four-point correlation functions of worldsheet primary operators are computed in this section by solving the relevant Knizhnik - Zamolodchikov (KZ) equations. As we will explain the resulting amplitudes are a simple generalization of the amplitudes of the $\mathbf{H}_4$ WZNW model. In section 6 the same results will be reproduced by resorting to the Wakimoto free-field representation. As in the previous section we find it convenient to first discuss the non-symmetric ($\mu_1 \neq \mu_2$) case and then pass to the symmetric ($\mu_1 = \mu_2$) case where $SU(2)_I$ invariance is needed in order to completely fix the correlators.

In general, world-sheet conformal invariance and global Ward identities allow us to write

$$G(z_i, \bar{z}_i, x_i, \bar{x}_i) = \prod_{i<j}^{4} |z_{ij}|^{2\left(\frac{h}{3}-h_i-h_j\right)} K(x_i, \bar{x}_i) \mathcal{G}(z, \bar{z}, x, \bar{x}) \ , \tag{5.0.1}$$

where $h = \sum_{i=1}^{4} h_i$ and the $SL(2, \mathbb{C})$ invariant cross-ratios z, $\bar{z}$ are defined according to

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}} \ , \qquad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} \ . \tag{5.0.2}$$

The form of the function K and the expression of the $\widehat{\mathcal{H}}_6$ invariants x in terms of the x_i are fixed by the global symmetries but are different for different types of correlators and therefore their explicit form will be given in the next sub-sections.

The four-point amplitudes are non trivial only when

$$L = -\sum_{i=1}^{4} \hat{\jmath}_i = \mu_1 q_1 + \mu_2 q_2 \ , \tag{5.0.3}$$

for some integers q_α. In the generic case for a given L these integers are uniquely fixed and the Ward identities fix the form of the functions K up to a function of two $\mathcal{H}_6$ invariants[4] x_1 and x_2. The KZ equations can be schematically written in the following form

$$\partial_z \mathcal{G}(z, x_1, x_2) = \sum_{\alpha=1}^{2} D_{\mathcal{H}_4, q_\alpha}(z, x_\alpha) \mathcal{G}(z, x_1, x_2) \ , \tag{5.0.4}$$

where the $D_{\mathcal{H}_4, q_\alpha}$ are differential operators closely related to those that appear in the KZ equations for the NW model based on the $\widehat{\mathcal{H}}_4$ affine algebra [1]. The equations are therefore easily solved by setting

$$\mathcal{G}_{q_1,q_2}(z, x_1, x_2) = \mathcal{G}_{\mathcal{H}_4, q_1}(z, x_1) \mathcal{G}_{\mathcal{H}_4, q_2}(z, x_2) \ . \tag{5.0.5}$$

[4]Sometimes we will collectively denote the $\mathcal{H}_6$ invariants x_1 and x_2 by x_α with $\alpha = 1, 2$. They should not be confused with the components of the charge variables $x_{i\alpha}$ that carry an additional label associated to the insertion point $i = 1, ..., 4$.

When $\mu_1 = \mu_2$, there are several integers that satisfy (5.0.3) and the $SU(2)_I$ invariant correlators can be obtained by summing over all possible pairs (q_1, q_2) such that $(q_1 + q_2) = L/\mu = Q$

$$\mathcal{G}_Q(z, x_1, x_2) = \sum_{q_1=0}^{Q} \mathcal{G}_{\mathcal{H}_4,q_1}(z, x_1)\mathcal{G}_{\mathcal{H}_4,Q-q_1}(z, x_2) \ . \tag{5.0.6}$$

This is the same procedure we used for the three-point functions and reflects the existence of new couplings between states in $\hat{\mathcal{H}}_6$ representations at the enhanced symmetry point. In the following we will describe the various types of four-point correlation functions.

5.1 $\langle + + + - \rangle$ correlators

Consider a correlator of the form

$$G_{+++-} = \langle \Phi^+_{p_1,\hat{\jmath}_1} \Phi^+_{p_2,\hat{\jmath}_2} \Phi^+_{p_3,\hat{\jmath}_3} \Phi^-_{p_4,\hat{\jmath}_4} \rangle \ , \qquad p_1 + p_2 + p_3 = p_4 \ . \tag{5.1.1}$$

This is the simplest 'extremal' case. From the decomposition of the tensor products of $\mathcal{H}_6$ representations displayed in Eq. (4.1.6) it follows that the correlator vanishes for $L < 0$ while for $L \geq 0$, $L = \mu_1 q_1 + \mu_2 q_2$ it decomposes into the sum of a finite number $N = (q_1+1)(q_2+1)$ of conformal blocks which correspond to the propagation in the s-channel of the representations $\Phi^+_{p_1+p_2,\hat{\jmath}_1+\hat{\jmath}_2+\mu_1 n_1+\mu_2 n_2}$ with $n_1 = 0, ..., q_1$ and $n_2 = 0, ..., q_2$. Global $\mathcal{H}_6$ symmetry yields

$$K(q_1, q_2) = \prod_{\alpha=1}^{2} \left| e^{-\mu_\alpha x_4^\alpha (p_1 x_{1\alpha} + p_2 x_{2\alpha} + p_3 x_{3\alpha})} \right|^2 |x_{3\alpha} - x_{1\alpha}|^{2q_\alpha} \ , \tag{5.1.2}$$

up to a function of the two invariants $(\alpha = 1, 2)$

$$x_\alpha = \frac{x_{2\alpha} - x_{1\alpha}}{x_{3\alpha} - x_{1\alpha}} \ . \tag{5.1.3}$$

We decompose the amplitude in a sum over the conformal blocks and write

$$\mathcal{G}_{q_1,q_2}(z, \bar{z}, x_\alpha, \bar{x}^\alpha) \sim \sum_{n_1=0}^{q_1} \sum_{n_2=0}^{q_2} \mathcal{F}_{n_1,n_2}(z, x_\alpha) \bar{\mathcal{F}}_{n_1,n_2}(\bar{z}, \bar{x}^\alpha) \ . \tag{5.1.4}$$

We set $\mathcal{F}_{n_1,n_2} = z^{\kappa_{12}}(1 - z)^{\kappa_{14}} F_{n_1,n_2}$ where

$$\begin{aligned} \kappa_{12} &= h_1 + h_2 - \frac{h}{3} - \hat{\jmath}_2 p_1 - \hat{\jmath}_1 p_2 - (\mu_1^2 + \mu_2^2) p_1 p_2 \ , \\ \kappa_{14} &= h_1 + h_4 - \frac{h}{3} - \hat{\jmath}_4 p_1 + \hat{\jmath}_1 p_4 + (\mu_1^2 + \mu_2^2) p_1 p_4 - (\mu_1 + \mu_2) p_1 + L(p_2 + p_3) \ , \end{aligned} \tag{5.1.5}$$

and where the F_{n_1,n_2} satisfy the following KZ equation

$$\partial_z F_{n_1,n_2}(z,x_1,x_2) = \frac{1}{z}\sum_{\alpha=1}^{2}\mu_\alpha\left[-(p_1x_\alpha + p_2x_\alpha(1-x_\alpha))\partial_{x_\alpha} + q_\alpha p_2 x_\alpha\right]F_{n_1,n_2}(z,x_1,x_2)$$

$$-\frac{1}{1-z}\sum_{\alpha=1}^{2}\mu_\alpha\left[(1-x_\alpha)(p_2x_\alpha+p_3)\partial_{x_\alpha} + q_\alpha p_2(1-x_\alpha)\right]F_{n_1,n_2}(z,x_1,x_2)\ . \qquad (5.1.6)$$

The explicit form of the conformal blocks is

$$F_{n_1,n_2}(z,x_1,x_2) = \prod_{\alpha=1}^{2} f(\mu_\alpha,z,x_\alpha)^{n_\alpha} g(\mu_\alpha,z,x_\alpha)^{q_\alpha-n_\alpha}\ , \qquad n_\alpha = 0,...,q_\alpha\ . \qquad (5.1.7)$$

Here

$$\begin{aligned} f(\mu_\alpha,z,x_\alpha) &= \frac{\mu_\alpha p_3}{1-\mu_\alpha(p_1+p_2)} z^{1-\mu_\alpha(p_1+p_2)}\varphi_0(\mu_\alpha) - x_\alpha z^{-\mu_\alpha(p_1+p_2)}\varphi_1(\mu_\alpha)\ , \\ g(\mu_\alpha,z,x_\alpha) &= \gamma_0(\mu_\alpha) - \frac{x_\alpha p_2}{p_1+p_2}\gamma_1(\mu_\alpha)\ , \end{aligned} \qquad (5.1.8)$$

and

$$\begin{aligned} \varphi_0(\mu) &= F(1-\mu p_1, 1+\mu p_3, 2-\mu p_1-\mu p_2, z)\ , \qquad \varphi_1(\mu) = F(1-\mu p_1, \mu p_3, 1-\mu p_1-\mu p_2, z)\ , \\ \gamma_0(\mu) &= F(\mu p_2, \mu p_4, \mu p_1+\mu p_2, z)\ , \qquad \gamma_1(\mu) = F(1+\mu p_2, \mu p_4, 1+\mu p_1+\mu p_2, z)\ , \end{aligned} \qquad (5.1.9)$$

where $F(a,b,c,z)$ is the standard ${}_1F_2$ hypergeometric function.

We can now reconstruct the four-point function as a monodromy invariant combination of the conformal blocks and the result is

$$\mathcal{G}_{q_1,q_2}(z,\bar z,x_\alpha,\bar x^\alpha) = |z|^{2\kappa_{12}}|1-z|^{2\kappa_{14}}\prod_{\alpha=1}^{2}\frac{\sqrt{\tau(\mu_\alpha)}}{q_\alpha!}\left[C_{12}(\mu_\alpha)|f(\mu_\alpha,z,x_\alpha)|^2 + C_{34}(\mu_\alpha)|g(\mu_\alpha,z,x_\alpha)|^2\right]^{q_\alpha}\ , \qquad (5.1.10)$$

where $\tau(\mu) = C_{12}(\mu)C_{34}(\mu)$ and

$$C_{12}(\mu) = \frac{\gamma(\mu(p_1+p_2))}{\gamma(\mu p_1)\gamma(\mu p_2)}\ , \qquad C_{34}(\mu) = \frac{\gamma(\mu p_4)}{\gamma(\mu p_3)\gamma(\mu(p_4-p_3))}\ . \qquad (5.1.11)$$

When $\mu_1 = \mu_2 = \mu$ we set $Q = L/\mu = \sum_\alpha q_\alpha$ and find the $SU(2)_I$ invariant combination

$$\begin{aligned} K_Q(x_\alpha,\bar x^\alpha)\mathcal{G}_Q(z,\bar z,x_\alpha,\bar x^\alpha) &= \sum_{q_1=0}^{Q} K(q_1,Q-q_1)\mathcal{G}_{q_1,Q-q_1}(z,\bar z,x_\alpha,\bar x^\alpha) = \\ = \quad & |z|^{2\kappa_{12}}|1-z|^{2\kappa_{14}} \times \times \prod_{\alpha=1}^{2}\left|e^{-\mu_\alpha x_4^\alpha(p_1x_{1\alpha}+p_2x_{2\alpha}+p_3x_{3\alpha})}\right|^2\frac{\tau(\mu)}{Q!}\times \\ & \times\left[\sum_{\alpha=1}^{2}\left(C_{12}(\mu)|x_{13\alpha}f(\mu,z,x_\alpha)|^2 + C_{34}(\mu)|x_{13\alpha}g(\mu,z,x_\alpha)|^2\right)\right]^Q\ . \end{aligned} \qquad (5.1.12)$$

5.2 $\langle + - + - \rangle$ correlators

The next class of correlators we want to discuss is of the following form

$$G_{+-+-} = \langle \Phi^{+}_{p_1,\hat{\jmath}_1} \Phi^{-}_{p_2,\hat{\jmath}_2} \Phi^{+}_{p_3,\hat{\jmath}_3} \Phi^{-}_{p_4,\hat{\jmath}_4} \rangle \ , \qquad p_1 + p_3 = p_2 + p_4 \ , \tag{5.2.1}$$

and also in this case we write $L = -\sum_i \hat{\jmath}_i = \sum_\alpha \mu_\alpha q_\alpha$. The Ward identities give

$$K(q_1,q_2) = \prod_{\alpha=1}^{2} \left| e^{-\mu_\alpha p_2 x_{1\alpha} x_2^\alpha - \mu_\alpha p_3 x_{3\alpha} x_4^\alpha - \mu_\alpha (p_1 - p_2) x_{1\alpha} x_4^\alpha} (x_{1\alpha} - x_{3\alpha})^{q_\alpha} \right|^2 \ , \tag{5.2.2}$$

and the two invariants (no sum over $\alpha = 1,2$)

$$x_\alpha = (x_{1\alpha} - x_{3\alpha})(x_2^\alpha - x_4^\alpha) \ . \tag{5.2.3}$$

We pass to the conformal blocks and set $\mathcal{F}_{n_1,n_2} = z^{\kappa_{12}}(1-z)^{\kappa_{14}} F_{n_1,n_2}$ where

$$\begin{aligned} \kappa_{12} &= h_1 + h_2 - \frac{h}{3} + (\mu_1^2 + \mu_2^2) p_1 p_2 - \hat{\jmath}_2 p_1 + \hat{\jmath}_1 p_2 - (\mu_1 + \mu_2) p_2 \ , \\ \kappa_{14} &= h_1 + h_4 - \frac{h}{3} + (\mu_1^2 + \mu_2^2) p_1 p_4 - \hat{\jmath}_4 p_1 + \hat{\jmath}_1 p_4 - (\mu_1 + \mu_2) p_4 \ . \end{aligned} \tag{5.2.4}$$

The F_{n_1,n_2} solve the following KZ equation

$$\begin{aligned} z(1-z)\partial_z F_{n_1,n_2}(z,x_1,x_2) &= z \sum_{\alpha=1}^{2} \Big[-2a_\alpha x_\alpha \partial_{x_\alpha} + \frac{x_\alpha}{4}(b_\alpha^2 - c_\alpha^2) - \\ - \ \rho_{\alpha 12} - \rho_{\alpha 14} \Big] F_{n_1,n_2}(z,x_1,x_2) &+ \sum_{\alpha=1}^{2} \Big[x_\alpha \partial^2_{x_\alpha} + (a_\alpha x_\alpha + 1 + q_\alpha)\,\partial_{x_\alpha} \\ + \ \frac{x_\alpha}{4}(a_\alpha^2 - b_\alpha^2) + \rho_{\alpha 12} \Big] F_{n_1,n_2}(z,x_1,x_2) \ , \end{aligned} \tag{5.2.5}$$

with

$$\rho_{\alpha 12} = \frac{(1+q_\alpha)}{2}(a_\alpha - b_\alpha) \ , \qquad \rho_{\alpha 14} = \frac{(1+q_\alpha)}{2}(a_\alpha - c_\alpha) \ , \tag{5.2.6}$$

and

$$2a_\alpha = \mu_\alpha(p_1 + p_3) \ , \quad b_\alpha = \mu_\alpha(p_1 - p_2) \ , \quad c_\alpha = \mu_\alpha(p_2 - p_3) \ . \tag{5.2.7}$$

The conformal blocks are very similar to the conformal blocks for the $\mathbf{H}_4$ WZNW model [1]

$$F_{n_1,n_2}(z,x_1,x_2) = \prod_{\alpha=1}^{2} \nu_{n_\alpha} \frac{e^{\mu_\alpha x_\alpha z p_3 - z(1-z)\mu_\alpha \partial \ln f_1(\mu_\alpha,z)}}{(f_1(\mu_\alpha,z))^{1+q_\alpha}} L^{q_\alpha}_{n_\alpha}[x_\alpha g(\mu_\alpha,z)] \left(\frac{f_2(\mu_\alpha,z)}{f_1(\mu_\alpha,z)} \right)^{n_\alpha} \ , \tag{5.2.8}$$

where $n_\alpha \in \mathbb{N}$ and L_n^q is the n-th generalized Laguerre polynomial. We also introduced the functions

$$\begin{aligned} f_1(\mu, z) &= F(\mu p_3, 1-\mu p_1, 1-\mu p_1+\mu p_2, z) \ , \\ f_2(\mu, z) &= z^{\mu(p_1-p_2)} F(\mu p_4, 1-\mu p_2, 1-\mu p_2+\mu p_1, z) \ , \end{aligned} \tag{5.2.9}$$

and

$$g = -z(1-z)\partial \ln\left(f_2/f_1\right) \ , \qquad \nu_{n_\alpha} = \frac{n_\alpha!}{[\mu_\alpha(p_1-p_2)]^{n_\alpha}} \ . \tag{5.2.10}$$

The four-point correlator can be written in a compact form using the combination

$$S(\mu_\alpha, z, \bar{z}) = |f_1(\mu_\alpha, z)|^2 - \rho(\mu_\alpha)|f_2(\mu_\alpha, z)|^2 \ , \qquad \rho(\mu) = \frac{\tilde{C}_{12}(\mu)\tilde{C}_{34}(\mu)}{\mu^2(p_1-p_2)^2} \ , \tag{5.2.11}$$

where we defined

$$\tilde{C}_{12}(\mu) = \frac{\gamma(\mu p_1)}{\gamma(\mu p_2)\gamma(\mu(p_1-p_2))} \ , \qquad \tilde{C}_{34}(\mu) = \frac{\gamma(\mu p_4)}{\gamma(\mu p_3)\gamma(\mu(p_4-p_3))} \ . \tag{5.2.12}$$

The four-point function reads

$$\begin{aligned} \mathcal{G}_{q_1,q_2}(z, \bar{z}, x_\alpha, \bar{x}^\alpha) = |z|^{2\kappa_{12}}|1-z|^{2\kappa_{14}} \prod_{\alpha=1}^{2} \frac{\tau(\mu_\alpha, q_\alpha)}{S(\mu_\alpha, z)} \left| e^{\mu_\alpha p_3 x_\alpha z - x_\alpha z(1-z)\partial_z \ln S(\mu_\alpha, z)} \right|^2 \times \\ \times |x_\alpha z^{b_\alpha}(1-z)^{c_\alpha}|^{-q_\alpha} I_{q_\alpha}(\zeta_\alpha) \ , \end{aligned} \tag{5.2.13}$$

where $I_q(\zeta)$ is a modified Bessel function and

$$\zeta_\alpha = \frac{2\sqrt{\rho(\mu_a)}|\mu_\alpha(p_1-p_2)x_\alpha z^{b_\alpha}(1-z)^{c_\alpha}|}{S(\mu_\alpha, z)} \ , \qquad \tau(\mu, q) = \tilde{C}_{12}(\mu)^{\frac{1-q}{2}} \tilde{C}_{34}(\mu)^{\frac{1+q}{2}} \ . \tag{5.2.14}$$

When $\mu_1 = \mu_2 = \mu$ the $SU(2)_I$ invariant correlator is given by the sum over $q_1 \in \mathbb{Z}$ with $q_2 = Q - q_1$ and $Q = L/\mu$. The addition formula for Bessel functions leads to

$$\mathcal{G}_Q(z, \bar{z}, x_\alpha, \bar{x}^\alpha) = \frac{\tau(\mu, Q)|z|^{2\kappa_{12}-bQ}|1-z|^{2\kappa_{14}-cQ}}{S(\mu, z)^2} \frac{||x_{13}||^Q}{||x_{24}||^Q} \left| e^{xz[\mu p_3 - (1-z)\partial_z \ln S(\mu, z)]} \right|^2 I_Q(\zeta) \ , \tag{5.2.15}$$

where

$$\zeta = \frac{2\sqrt{C_{12}C_{34}}|z^b(1-z)^c|}{S(\mu, z)} ||x_{13}|| ||x_{24}|| \ , \tag{5.2.16}$$

and $x = x_{13} \cdot x_{24} = \sum_\alpha (x_{1\alpha} - x_{3\alpha})(x_2^\alpha - x_4^\alpha)$ as well as $||x_{ij}||^2 = \sum_\alpha |x_{i\alpha} - x_{j\alpha}|^2$ are $SU(2)_I$ invariant.

The factorization properties of these correlators can be analyzed following [1]. In this way one can check that the modified highest weight representations introduced in section 3 actually appear in the intermediate channels.

5.3 $\langle ++-\,0\rangle$ correlators

Let us describe now a correlator of the form

$$G_{++-0} = \langle \Phi^+_{p_1,\hat{j}_1} \Phi^+_{p_2,\hat{j}_2} \Phi^-_{p_3,\hat{j}_3} \Phi^0_{s_1,s_2,\hat{j}_4} \rangle \ , \qquad p_1 + p_2 = p_3 \ . \tag{5.3.1}$$

From the global symmetry constraints we derive

$$K(q_1,q_2) = \prod_{\alpha=1}^{2} \left| e^{-\mu_\alpha x_3^\alpha (p_1 x_{1\alpha} + p_2 x_{2\alpha}) - \frac{s_\alpha}{\sqrt{2}} x_3^\alpha x_{4\alpha} - \frac{s_\alpha}{2\sqrt{2}} (x_{1\alpha}+x_{2\alpha}) x_4^\alpha} x_{4\alpha}^{q_\alpha} \right|^2 \ , \tag{5.3.2}$$

up to a function of the two invariants (no sum over $\alpha = 1,2$)

$$x_\alpha = (x_{1\alpha} - x_{2\alpha}) x_4^\alpha \ . \tag{5.3.3}$$

We rewrite the conformal blocks as

$$\mathcal{F}_{n_1,n_2} = z^{\kappa_{12}} (1-z)^{\kappa_{14}} F_{n_1,n_2} \ , \tag{5.3.4}$$

where

$$\begin{aligned} \kappa_{12} &= h_1 + h_2 - \frac{h}{3} - p_1 \hat{j}_2 - p_2 \hat{j}_1 - (\mu_1^2 + \mu_2^2) p_1 p_2 \ , \\ \kappa_{14} &= h_1 + h_4 - \frac{h}{3} - p_1 \hat{j}_4 - L p_1 - \frac{s_1^2 + s_2^2}{4} \ . \end{aligned} \tag{5.3.5}$$

The KZ equation then reads

$$\begin{aligned} z(1-z)\partial_z F_{n_1,n_2}(z,x_1,x_2) &= -\sum_{\alpha=1}^{2} \left[\mu_\alpha p_3 x_\alpha \partial_{x_\alpha} + \frac{s_\alpha}{2\sqrt{2}} \mu_\alpha (p_1 - p_2) x_\alpha \right] F_{n_1,n_2}(z,x_1,x_2) \\ &+ z \sum_{\alpha=1}^{2} \left[\left(\mu_\alpha p_2 x_\alpha - \frac{s_\alpha}{\sqrt{2}} \right) \partial_{x_\alpha} - \frac{s_\alpha \mu_\alpha p_2}{2\sqrt{2}} x_\alpha \right] F_{n_1,n_2}(z,x_1,x_2) \ , \end{aligned} \tag{5.3.6}$$

and the solution is

$$F_{n_1,n_2}(z,x_\alpha) = \prod_{\alpha=1}^{2} [s_\alpha \varphi(\mu_\alpha,z) + x_\alpha \omega(\mu_\alpha,z)]^{n_\alpha} e^{s_\alpha^2 \eta(\mu_\alpha,z) + s_\alpha x_\alpha \chi(\mu_\alpha,z)} \ , \tag{5.3.7}$$

with $n_1, n_2 \geq 0$. We have introduced the following functions

$$\begin{aligned} \varphi(\mu,z) &= \frac{z^{1-\mu p_3}}{\sqrt{2}(1-\mu p_3)} F(1-\mu p_1, 1-\mu p_3, 2-\mu p_3, z) \ , \\ \omega(\mu,z) &= -z^{-\mu p_3} (1-z)^{\mu p_1} \ , \\ \chi(\mu,z) &= -\frac{1}{2\sqrt{2}} + \frac{p_2}{\sqrt{2} p_3} (1-z) F(1+\mu p_2, 1, 1+\mu p_3, z) \ , \\ \eta(\mu,z) &= -\frac{z p_2}{2 p_3} \, {}_3F_2(1+\mu p_2, 1, 1; 1+\mu p_3, 2; z) - \frac{1}{4} \ln(1-z) \ . \end{aligned} \tag{5.3.8}$$

The four-point function is then given by

$$\mathcal{G}_{q_1,q_2}(z,\bar{z},x_\alpha,\bar{x}^\alpha) = |z|^{2\kappa_{12}}|1-z|^{2\kappa_{14}} \prod_{\alpha=1}^{2} C_{12}^{1/2}(\mu_\alpha) C_{+-0}(\mu_\alpha,p_3,s_\alpha) \times \\ e^{C_{12}(\mu_\alpha)|s_\alpha\varphi(\mu_\alpha,z)+x_\alpha\omega(\mu_\alpha,z)|^2} \left| e^{s_\alpha^2\eta(\mu_\alpha,z)+s_\alpha x_\alpha\chi(\mu_\alpha,z)} \right|^2 \ , \quad (5.3.9)$$

where

$$C_{12}(\mu) = \frac{\gamma(\mu(p_1+p_2))}{\gamma(\mu p_1)\gamma(\mu p_2)} \ , \qquad C_{+-0}(\mu,p_3,s) = e^{\frac{s^2}{2}[\psi(\mu p_3)+\psi(1-\mu p_3)-2\psi(1)]} \ . \qquad (5.3.10)$$

The $SU(2)_I$ invariant correlator at the point $\mu_1=\mu_2=\mu$ is obtained after summing over $q_1 \in \mathbb{Z}$ with $q_1+q_2=Q=L/\mu$.

5.4 $\langle + - \ 0 \ 0\rangle$ correlators

The last correlator we have to consider is of the form

$$\langle \Phi^+_{p,\hat{j}_1} \Phi^-_{p,\hat{j}_2} \Phi^0_{s_{3\alpha},\hat{j}_3} \Phi^0_{s_{4\alpha},\hat{j}_4} \rangle \ , \quad p_1 = p_2. \qquad (5.4.1)$$

The Ward identities give

$$K(q_1,q_2) = \prod_{\alpha=1}^{2} \left| e^{-\mu_\alpha p x_{1\alpha} x_2^\alpha - \frac{x_{1\alpha}}{\sqrt{2}}\left(s_{3\alpha}x_3^\alpha + s_{4\alpha}x_4^\alpha\right) - \frac{x_2^\alpha}{\sqrt{2}}(s_{3\alpha}x_{3\alpha}+s_{4\alpha}x_{4\alpha})} x_{3\alpha}^{q_\alpha} \right|^2 \ , \qquad (5.4.2)$$

up to a function of the two invariants (no sum over $\alpha = 1,2$) $x_\alpha = x_3^\alpha x_{4\alpha}$.

We decompose this correlator around $z=1$ setting $u=1-z$, since the conformal blocks turn out to be simpler and rewrite them as

$$\mathcal{F}_{n_1,n_2} = z^{\kappa_{12}}(1-z)^{\kappa_{14}} F_{n_1,n_2} \ , \qquad (5.4.3)$$

where

$$\kappa_{14} = h_1 + h_4 - \frac{h}{3} - p\hat{j}_4 - \sum_{\alpha=1}^{2} \frac{s_{4\alpha}^2}{2} \ , \qquad \kappa_{12} = \sum_{\alpha=1}^{2} \frac{s_{3\alpha}^2 + s_{4\alpha}^2}{2} - \frac{h}{3} \ . \qquad (5.4.4)$$

The KZ equation

$$\partial_u F_{n_1,n_2}(z,x_1,x_2) = -\frac{1}{u}\sum_{\alpha=1}^{2}\left[\mu_\alpha p x_\alpha \partial_{x_\alpha} + \frac{s_{3\alpha}s_{4\alpha}x_\alpha}{2}\right] F_{n_1,n_2}(z,x_1,x_2) \\ - \frac{1}{1-u}\sum_{\alpha=1}^{2}\frac{s_{3\alpha}s_{4\alpha}}{2}\left(x_\alpha + \frac{1}{x_\alpha}\right) F_{n_1,n_2}(z,x_1,x_2) \ , \qquad (5.4.5)$$

has the solutions

$$F_{n_1,n_2}(u,x_\alpha) = \prod_{\alpha=1}^{2} (x_\alpha u^{-\mu_\alpha p})^{n_\alpha} e^{x_\alpha \omega(\mu_\alpha,u)+x^\alpha\chi(\mu_\alpha,u)} \ , \qquad (5.4.6)$$

with $n_1, n_2 \in \mathbb{Z}$, $x^\alpha = x_{3\alpha} x_4^\alpha = 1/x_\alpha$ and

$$\omega(\mu, u) = -\frac{s_3 s_4}{2\mu p}\, F(\mu p, 1, 1+\mu p, u)\,, \quad \chi(\mu, u) = -\frac{s_3 s_4}{2(1-\mu p)}\, u\, F(1-\mu p, 1, 2-\mu p, u)\,. \tag{5.4.7}$$

The four-point function is then given by

$$\mathcal{G}(u, \bar{u}, x_\alpha, \bar{x}^\alpha) = |u|^{2\kappa_{12}} |1-u|^{2\kappa_{14}} \prod_{\alpha=1}^{2} \tau(\mu_\alpha) \left| e^{x_\alpha \omega(\mu_\alpha, u) + x^\alpha \chi(\mu_\alpha, u)} \right|^2 \sum_{n_\alpha \in \mathbb{Z}} \left| x_\alpha u^{-\mu_\alpha p} \right|^{2n_\alpha} \,, \tag{5.4.8}$$

where $\tau(\mu) = C_{+-0}(\mu, p, s_3) C_{+-0}(\mu, p, s_4)$

The $SU(2)_I$ invariant correlator at the point $\mu_1 = \mu_2 = \mu$ is obtained after summing over $q_1 \in \mathbb{Z}$ with $q_1 + q_2 = Q = L/\mu$.

6. Wakimoto representation

In this section we construct a free field representation for the $\widehat{\mathcal{H}}_6$ algebra starting from the standard Wakimoto realization for $\widehat{SL}(2, \mathbb{R})$ and $\widehat{SU}(2)$ [67] and contracting the currents of both CFTs as indicated in section 2. Then we use this approach to compute two, three and four-point correlators that only involve $\Phi^\pm$ vertex operators and reproduce the results obtained in the previous sections. This free field representation was introduced by Cheung, Freidel and Savvidy [29] and used to evaluate correlation functions for $\widehat{\mathcal{H}}_4$.

6.1 $\widehat{\mathcal{H}}_6$ free field representation

The Wakimoto representation of the $\widehat{SL}(2, \mathbb{R})$ current algebra requires a pair of commuting ghost fields $\beta_1(z)$ and $\gamma^1(z)$ (the index 1 here is a label) with propagator $\langle \beta_1(z) \gamma^1(w) \rangle = 1/(z-w)$, and a free boson $\phi(z)$ with $\langle \phi(z) \phi(w) \rangle = -\log(z-w)$. The $\widehat{SL}(2, \mathbb{R})$ currents can then be written as

$$\begin{aligned} K^+(z) &= -\beta_1 \,, \\ K^-(z) &= -\beta_1 \gamma^1 \gamma^1 + \alpha_+ \gamma^1 \partial\phi - k_1 \partial\gamma^1 \,, \\ K^3(z) &= -\beta_1 \gamma^1 + \frac{\alpha_+}{2} \partial\phi \,, \end{aligned} \tag{6.1.1}$$

where $\alpha_+^2 \equiv 2(k_1 - 2)$. Similarly for $\widehat{SU}(2)$ we introduce a second pair of ghost fields $\beta_2(z)$ and $\gamma^2(z)$ (here the index 2 is a label) with world-sheet propagator $\langle \beta_2(z) \gamma^2(w) \rangle = 1/(z-w)$, and a free boson $\varphi(z)$ with $\langle \varphi(z) \varphi(w) \rangle = -\log(z-w)$. The currents are then given by

$$\begin{aligned} J^+(z) &= -\beta_2 \,, \\ J^-(z) &= \beta_2 \gamma^2 \gamma^2 - i\alpha_- \gamma^2 \partial\varphi - k_2 \partial\gamma^2 \,, \\ J^3(z) &= -\beta_2 \gamma^2 + \frac{i\alpha_-}{2} \partial\varphi \,, \end{aligned} \tag{6.1.2}$$

where $\alpha_-^2 \equiv 2(k_2+2)$. In order to obtain a Wakimoto realization for the $\widehat{\mathcal{H}}_6$ algebra, we rescale the two ghost systems

$$\beta_\alpha \to \sqrt{\frac{k_\alpha}{2}}\,\beta_\alpha \;, \qquad \gamma^\alpha \to \sqrt{\frac{2}{k_\alpha}}\,\gamma^\alpha \;, \tag{6.1.3}$$

and introduce the light-cone fields u and v

$$\phi = -i\sqrt{\frac{k_1}{2}}\mu_1 u - \frac{i}{\sqrt{2k_1}}\frac{v}{\mu_1} \;, \qquad \varphi = \sqrt{\frac{k_2}{2}}\mu_2 u - \frac{1}{\sqrt{2k_2}}\frac{v}{\mu_2} \;, \tag{6.1.4}$$

with $u(z)v(w) \sim \ln(z-w)$. We then perform the current contraction as prescribed in (2.10), with the result

$$\begin{aligned} P_\alpha^+(z) &= -\beta_\alpha \;, \\ P^{-\alpha}(z) &= -2\partial\gamma^\alpha - 2i\mu_\alpha \partial u\,\gamma^\alpha \;, \\ J(z) &= i\,\mu_\alpha\beta_\alpha\gamma^\alpha - \partial v + \frac{\mu_1^2+\mu_2^2}{2}\partial u \;, \\ K(z) &= -\partial u. \end{aligned} \tag{6.1.5}$$

The $\hat{\mathcal{H}}_6$ stress-energy tensor follows from the limit of $T_{SL(2,\mathbb{R})}(z) + T_{SU(2)}(z)$ and is given by

$$T(z) = \sum_{\alpha=1}^{2} :\beta_\alpha(z)\partial\gamma^\alpha(z): + :\partial u(z)\,\partial v(z): - \frac{i}{2}(\mu_1+\mu_2)\partial^2 u \;, \tag{6.1.6}$$

where the last term appears when expressing the normal ordered product of the currents in terms of the Wakimoto fields.

The $\Phi^+_{p,\hat{\jmath}}$ primary vertex operators similarly follow from the $SL(2,\mathbb{R}) \times SU(2)$ primary vertex operators in the $\mathcal{D}_l^- \times V_{\tilde{l}}$ representation

$$V_{l,m;\tilde{l},\tilde{m}} = (-\gamma^1)^{-l-m}(-\gamma^2)^{\tilde{l}-\tilde{m}} e^{\frac{2\tilde{l}}{\alpha_+}\phi + \frac{2i\tilde{l}}{\alpha_-}\varphi} \;, \tag{6.1.7}$$

where m is the eigenvalue of K^3. Introducing the charge variables we can collect all the components in a single field

$$\Phi^+_{l,\tilde{l}}(z,x_\alpha) = \left(1 + x_1\gamma^1\right)^{-2l}\left(1 - x_2\gamma^2\right)^{2\tilde{l}} e^{\frac{2\tilde{l}}{\alpha_+}\phi + \frac{2\tilde{l}}{\alpha_-}i\varphi} \;, \tag{6.1.8}$$

that in the large k_1, k_2 limit becomes, after rescaling $x_\alpha \to \frac{x_\alpha}{\sqrt{k_\alpha}}$

$$\Phi^+_{p,\hat{\jmath}}(z,x_\alpha) = N(p,\hat{\jmath})e^{-\sqrt{2}\mu_\alpha p x_\alpha\gamma^\alpha - ipv - i\left(\hat{\jmath} + \frac{\mu_1^2+\mu_2^2}{2}p\right)u} \;. \tag{6.1.9}$$

It is easy to verify that this field satisfies the correct OPEs with the $\hat{\mathcal{H}}_6$ currents and that its conformal dimension is $h(p,\hat{\jmath}) = -p\hat{\jmath} + \frac{\mu_1 p}{2}(1-\mu_1 p) + \frac{\mu_2 p}{2}(1-\mu_2 p)$. If we

choose the normalization factor $N(p,\hat{\jmath}) = (\gamma(\mu_1 p)\gamma(\mu_2 p))^{-1/2}$ the vertex operators (6.1.9) precisely reproduce the results obtained in the previous sections.

The $\Phi^-_{p,\hat{\jmath}}$ vertex operators can be represented using an integral transform [29]

$$\Phi^-_{p,\hat{\jmath}}(z,x^\alpha) = \prod_{\alpha=1}^{2}\int d^2x_\alpha\,\gamma(\mu_a p)\frac{\mu_\alpha^2 p^2}{2\pi^2}\;e^{-\mu_\alpha p x_\alpha x^\alpha}\,\Phi^+_{-p,\hat{\jmath}+\mu_1+\mu_2}(z,x_\alpha)\ . \tag{6.1.10}$$

The Wakimoto representation can also be derived from the σ-model action written in the following form

$$S = \int\frac{d^2z}{2\pi}\left\{-\partial u\bar{\partial}v + \sum_{\alpha=1}^{2}\left[\beta_\alpha\bar{\partial}\gamma^\alpha + \bar{\beta}^\alpha\partial\bar{\gamma}_\alpha - \beta_\alpha\bar{\beta}^\alpha e^{-i\mu_\alpha u}\right]\right\}\ , \tag{6.1.11}$$

as we will review in appendix A. The non-chiral $SU(2)_I$ currents are

$$\mathcal{J}^a(z,\bar{z}) = i\,\gamma^\alpha(\sigma^a)_\alpha{}^\beta\,\beta_\beta\ , \qquad\qquad \bar{\mathcal{J}}^a(z,\bar{z}) = -i\,\bar{\beta}^\alpha(\sigma^a)_\alpha{}^\beta\,\bar{\gamma}_\beta\ . \tag{6.1.12}$$

Using the equations of motion

$$\beta_\alpha = e^{i\mu_\alpha u}\partial\bar{\gamma}_\alpha\ , \qquad\qquad \bar{\partial}\beta_\alpha = 0\ , \tag{6.1.13}$$

one can verify that they satisfy $\bar{\partial}\mathcal{J}^a + \partial\bar{\mathcal{J}}^a = 0$. Moreover their OPEs with the Wakimoto free fields are

$$\begin{aligned}
\mathcal{J}^a(z,\bar{z})\gamma^\alpha(z,\bar{z}) &\sim i\,\frac{\gamma^\beta(\sigma^a)_\beta{}^\alpha}{z-w}\ , &\qquad \bar{\mathcal{J}}^a(z,\bar{z})\bar{\gamma}_\alpha(z,\bar{z}) &\sim -i\,\frac{(\sigma^a)_\alpha{}^\beta\bar{\gamma}_\beta}{\bar{z}-\bar{w}}\ ,\\
\mathcal{J}^a(z,\bar{z})\beta_\alpha(z) &\sim -i\,\frac{(\sigma^a)_\alpha{}^\beta\beta_\beta}{z-w}\ , &\qquad \bar{\mathcal{J}}^a(z,\bar{z})\bar{\beta}^\alpha(\bar{z}) &\sim i\,\frac{\bar{\beta}^\beta(\sigma^a)_\beta{}^\alpha}{\bar{z}-\bar{w}}\ .
\end{aligned} \tag{6.1.14}$$

6.2 Correlators

In order to evaluate the correlation functions in this free-field approach, we first integrate over the zero modes of the Wakimoto fields using the invariant measure

$$\int du_0 dv_0\prod_{\alpha=1}^{2}d\gamma_0^\alpha\,d\bar{\gamma}_0^\alpha e^{i\mu_\alpha u_0}\ . \tag{6.2.1}$$

The presence of the interaction term

$$S_I = \sum_{\alpha=1}^{2}S_{I\alpha} = -\sum_{\alpha=1}^{2}\int\frac{d^2w}{2\pi}\;\beta_\alpha(w)\bar{\beta}^\alpha(\bar{w})e^{-i\mu_\alpha u(w,\bar{w})}\ , \tag{6.2.2}$$

in the action (6.1.11) leads to the insertion in the free field correlators of the screening operators

$$\sum_{q_1,q_2=0}^{\infty}\prod_{\alpha=1}^{2}\frac{1}{q_\alpha!}\left(\int\frac{d^2w_\alpha}{2\pi}\beta_\alpha\bar{\beta}^\alpha e^{-i\mu_\alpha u}\right)^{q_\alpha}\ . \tag{6.2.3}$$

Negative powers of the screening operator are needed in order to get sensible results for n-point correlation functions other than the 'extremal' ones, that only involve one $\Phi^-_{p_n,\hat{\jmath}_n}$ vertex operator and $n-1$ $\Phi^+_{p_i,\hat{\jmath}_i}$ vertex operators. This means that the sum over q_α should effectively runs over all integers, $q_\alpha \in \mathbb{Z}$, not only the positive ones. An 'extremal' n-point function can be written as

$$\sum_{q_1,q_2=0}^{\infty} \prod_{\alpha=1}^{2} \frac{1}{m_\alpha!} \left\langle \prod_{i=1}^{n-1} \Phi^+_{p_i,\hat{\jmath}_i}(z_i,\bar{z}_i,x_{i\alpha},\bar{x}^\alpha_i) \Phi^+_{-p_4,\hat{\jmath}_4+\mu_1+\mu_2}(z_n,\bar{z}_n,x_{n\alpha},\bar{x}^\alpha_n)\, S^{q_\alpha}_{I\alpha} \right\rangle \tag{6.2.4}$$

$$= \delta\left(\sum_i p_i - p_n\right) \prod_{i<j\neq 4} |z_i - z_j|^{-2p_i(\hat{\jmath}_j+\eta\, p_j)-2p_j(\hat{\jmath}_i+\eta\, p_i)} \prod_{i\neq n} |z_i - z_n|^{2p_n(\hat{\jmath}_i+\eta\, p_i)-2p_i(\hat{\jmath}_n-\eta\, p_n+\mu_1+\mu_2)}$$

$$\sum_{q_1,q_2=0}^{\infty} \delta\left(L - \mu_1 q_1 - \mu_2 q_2 \right) \prod_{\alpha=1}^{2} R(\mu_\alpha) \left| e^{-\mu_\alpha x^\alpha_n \sum_{i=1}^{n-1} p_i x_{i\alpha}} \right|^2 \frac{1}{q_\alpha!} \left(-2\mu_\alpha^2 \mathcal{I}_{\alpha,n}\right)^{q_\alpha} \ ,$$

where $L = -\sum_{i=1}^{n} \hat{\jmath}_i$, $\eta = \frac{\mu_1^2+\mu_2^2}{2}$ and

$$\mathcal{I}_{\alpha,n} = \int \frac{d^2w}{2\pi} \prod_{i=1}^{n-1} |z_i - w|^{-2\mu_\alpha p_i} |z_n - w|^{2\mu_\alpha p_n} \left| \sum_{i=1}^{n-1} \frac{p_i x_{i\alpha}}{w - z_i} - \frac{p_n x_{n\alpha}}{w - z_n} \right|^2 \ , \tag{6.2.5}$$

with the constraint $p_n x_{n\alpha} = \sum_{i=1}^{n-1} p_i x_{i\alpha}$. Finally the constant $R(\mu)$, related to the normalization of the operators in (6.1.9), is given by

$$R^2(\mu) = \frac{\gamma(\mu p_n)}{\prod_{i=1}^{n-1} \gamma(\mu p_i)} \ . \tag{6.2.6}$$

In (6.2.4) the two δ-functions arise from the integration over u_0 and v_0. Similarly the integration over the $\gamma_{0\alpha}$ leads to four other δ-functions that constrain the integration over the $x_{n\alpha}$ variables and give the exponential term. The other terms in (6.2.4) follow from the contraction of the free Wakimoto fields. Note that due to the second δ-function in (6.2.4) the correlator is non vanishing only when $L = \mu_1 q_1 + \mu_2 q_2$ where $q_\alpha \in \mathbb{N}$. Therefore the same structure we found before using current algebra techniques appears: for the generic background $\mu_1 \neq \mu_2$ only one term from the double sum in (6.2.4) contributes while for the $SU(2)_I$ invariant wave we have to add several terms. Let us consider some examples. We will need the following integral [55]

$$\int d^2t |t-z|^{2(c-b-1)} |t|^{2(b-1)} |t-1|^{-2a} = \frac{\pi\gamma(b)\gamma(c-b)}{\gamma(c)} |z|^{2(c-1)} |F(a,b,c;z)|^2$$

$$- \frac{\pi\gamma(c)\gamma(1+a-c)}{(1-c)^2\gamma(a)} |F(1+a-c,1+b-c,2-c;z)|^2 \ . \tag{6.2.7}$$

It follows from the general expression (6.2.4) that the two-point function $\langle +- \rangle$ coincides with (4.1.3), since only the $q_\alpha = 0$ terms are non-vanishing. For the $\langle ++- \rangle$

three-point coupling the integral (6.2.5) gives

$$-2\,\mu_\alpha^2\,\mathcal{I}_{\alpha,3} = |z_{12}|^{-2\mu_\alpha p_3}|z_{23}|^{2\mu_\alpha p_1}|z_{13}|^{2\mu_\alpha p_2}\frac{\gamma(\mu_\alpha p_3)}{\gamma(\mu_\alpha p_1)\gamma(\mu_\alpha p_2)}\,|x_{1\alpha}-x_{2\alpha}|^2\ , \qquad (6.2.8)$$

and the result precisely agrees with (4.1.8), (4.1.9). When $\mu_1=\mu_2$ the sum over q_α reconstructs the $SU(2)_I$ invariant coupling (4.1.14).

The four-point function $\langle +++-\rangle$ can be evaluated in a similar way. In this case

$$\begin{aligned} -2\,\mu_\alpha^2\,\mathcal{I}_{\alpha,4} = {} & |z_{12}|^{-2\mu_\alpha p_4}|z_{14}|^{-2\mu_\alpha(p_1-p_4)}|z_{34}|^{-2\mu_\alpha(p_3-p_4)}|z_{24}|^{2\mu_\alpha(p_2-p_4)} \\ & \left[C_{12}(\mu_\alpha)\,|x_{31\alpha}f(\mu_\alpha,x_\alpha,z)|^2 + C_{34}(\mu_\alpha)\,|x_{31\alpha}g(\mu_\alpha,x_\alpha,z)|^2\right]\ , \end{aligned} \qquad (6.2.9)$$

where the functions f and g are as defined in (5.1.8) and

$$C_{12}(\mu) = \frac{\gamma(\mu(p_1+p_2))}{\gamma(\mu p_1)\gamma(\mu p_2)}\ , \qquad C_{34}(\mu) = \frac{\gamma(\mu p_4)}{\gamma(\mu p_3)\gamma(\mu(p_4-p_3))}\ . \qquad (6.2.10)$$

We find again complete agreement with (5.1.10).

Finally the correlator $\langle +-+-\rangle$ can be obtained from the $\langle +++-\rangle$ correlator performing the integral transform (6.1.10) of the vertex operator inserted in z_2 [29], that is we send $(p_2,\hat{\jmath}_2)\to(-p_2,\hat{\jmath}_2+\mu_1+\mu_2)$ and evaluate the $x_{2\alpha}$ integral. We first rewrite

$$\begin{aligned} \mathcal{T} \equiv {} & \int d^2x_{2\alpha}\frac{\left|e^{-\mu_\alpha p_2 x_{24}^\alpha}\right|^2}{\Gamma(q_\alpha+1)}\left[C_{12}(\mu_\alpha)\,|x_{31\alpha}f(\mu_\alpha,x_\alpha,z)|^2 + C_{34}(\mu_\alpha)\,|x_{31\alpha}g(\mu_\alpha,x_\alpha,z)|^2\right]^{q_\alpha} \\ = {} & \int d^2x_{2\alpha}\frac{\left|e^{-\mu_\alpha p_2 x_{24}^\alpha}\right|^2}{\Gamma(q_\alpha+1)}\left[Ax_{2\alpha}\bar{x}_{2\alpha}+B\bar{x}_{2\alpha}+\bar{B}x_{2\alpha}+E\right]^{q_\alpha}\ , \end{aligned} \qquad (6.2.11)$$

and then evaluate the integral using

$$\int d^2u\,\left|e^{-u}u^t\right|^2 = \pi(-1)^{-1-t}\gamma(1+t)\ , \qquad (6.2.12)$$

which is a limit of (6.2.7). The result is

$$\mathcal{T} = \left|e^{\mu_\alpha p_2 x_{24}^\alpha \frac{B_\alpha}{A}}\right|^2\frac{|x_{24}^\alpha|^{-q_\alpha}}{2A}\left(\frac{B\bar{B}-EA}{\mu_\alpha^2p_2^2}\right)^{\frac{q_\alpha}{2}} I_{q_\alpha}\left(2\mu_\alpha p_2\,|x_{24}^\alpha|\sqrt{\frac{B\bar{B}-EA}{A^2}}\right)\ , \qquad (6.2.13)$$

where I_{q_α} is a modified Bessel function of integer order and

$$\begin{aligned} \frac{\mu_\alpha p_2 x_{24}^\alpha B}{A} &= -\mu_\alpha p_2 x_{1\alpha}x_{24}^\alpha + \mu_\alpha p_3 x_{13\alpha}x_{24}^\alpha z - \mu_\alpha p_2 x_{13\alpha}x_{24}^\alpha z(1-z)\partial_z \ln S(\mu_\alpha,z,\bar z)\ , \\ A &= -\frac{\mu_\alpha^2p_2^2}{\tilde{C}_{12}}|z|^{-2\mu_\alpha(p_1-p_2)}S(\mu_\alpha,z,\bar z)\ , \qquad 2\mu_\alpha p_2\,|x_{24}^\alpha|\sqrt{\frac{B\bar{B}-EA}{A^2}} = \zeta_\alpha\ . \end{aligned} \qquad (6.2.14)$$

The functions and constants that appear on the left-hand side of the previous equations were defined in (5.2.9 – 5.2.12) and (5.2.14). Combining (6.2.13) with the rest of the $\langle +++-\rangle$ correlator we obtain the $\langle +-+-\rangle$ correlator and also in this case the result coincides with (5.2.13) when $\mu_1\neq\mu_2$ and with (5.2.15) when $\mu_1=\mu_2$.

7. String amplitudes

In this section we study the string amplitudes in the Hpp-wave. After combining the results of the previous sections with the ones for the internal CFT and for the world-sheet ghosts, one can easily extract irreducible vertices and decay rates in closed form. The world-sheet integrals needed for the computation of four-point scattering amplitudes of scalar (tachyon) vertex operators are not elementary and we only study the appropriate singularities and interpret them in terms of OPE. As mentioned in section 2 the Hpp-wave with $\widehat{\mathcal{H}}_6$ affine Heisenberg symmetry that emerges in the Penrose limit of $AdS_3 \times S^3$ should be combined with extra degrees of freedom in order to represent a consistent background for the bosonic string. Quite independently of the initial values of $k_{SL(2,\mathbb{R})} = k_1$ and $k_{SU(2)} = k_2$, one needs to combine the resulting CFT that has $c = 6$ with some internal CFT with $c = 20$. For definiteness let us suppose this internal CFT to correspond to flat space $\mathbf{R}^{20}$ or to a torus T^{20}, but this choice is by no means crucial in the following.

In a covariant approach, such as the one followed throughout the paper, string states correspond to BRS invariant vertex operators. As usual, negative norm states correspond to unphysical 'polarizations'. These are absent for the scalar (tachyon) vertex operators we have constructed in section 3. Let us focus on the left-movers. Starting from a 'standard' HW ($\mu_\alpha p < 1$ for $\alpha = 1, 2$) primary state $|\Psi\rangle$ of $\widehat{\mathcal{H}}_6$, the Virasoro constraints

$$L_n|\Psi\rangle = 0 \ , \qquad \text{for} \quad n > 0 \ , \tag{7.1}$$

together with

$$L_0|\Psi\rangle = |\Psi\rangle \ , \tag{7.2}$$

project the Hilbert space on positive norm states. The mass-shell condition becomes

$$h^a_{p,\hat{\jmath}} + h_{int} + N = 1 \ , \tag{7.3}$$

where N is the total level, h_{int} is the contribution of the internal CFT, *i.e.* $h_{int} = |\vec{p}|^2/2$ and for $p \neq 0$

$$h^{\pm}_{p,\hat{\jmath}} = \mp p\hat{\jmath} + \frac{1}{2}\sum_{\alpha=1}^{2} \mu_\alpha p(1 - \mu_\alpha p) \ , \tag{7.4}$$

while for $p = 0$,

$$h^0_{s,\hat{\jmath}} = \frac{1}{2}s^2 = \frac{1}{2}\sum_{\alpha=1}^{2} s_\alpha^2 \ . \tag{7.5}$$

Outside the range $\mu_\alpha p < 1$ one has to consider spectral flowed representations when $\mu_1 = \mu_2 = \mu$ or MHW representations when $\mu_1 \neq \mu_2$, as discussed in section 3.

Let us concentrate for simplicity on $\mu_1 = \mu_2 = \mu$ with enhanced (non-chiral) $SU(2)_I$ invariance. In this particular case, spectral flow yields states with

$$h^{\pm,w}_{p,\hat{\jmath}} = \mp\left(p + \frac{w}{\mu}\right)\hat{\jmath} + \mu p(1-\mu p) \mp w\lambda \ , \tag{7.6}$$

where $\lambda = n_- - n_+$ is the total 'helicity' and, for $p = 0$,

$$h^{0,w}_{s,\hat{\jmath}} = \frac{w}{\mu}\hat{\jmath} - \frac{1}{2}s^2 - w\lambda \ . \tag{7.7}$$

The physics is similar to the case of the NW background [1]: whenever μp reaches an integer value in string units, stringy effects become important and one has to resort to spectral flow in order to make sense of the resulting state [28]. The string feels no confining potential and is free to move along the 'magnetized planes'. The analysis of AdS_3 leads qualitatively to the same conclusions [8]. Spectral flowed states can appear both in intermediate channels and as external legs. Even though in this paper we have only considered correlation functions with states in highest weight representations with $\mu|p| < 1$ as external legs, it is not difficult to generalize our results to include spectral flowed external states along the lines of [1].

In order to compute covariant string amplitudes in the Hpp-wave one has to combine the correlators computed in sections 4, 5 and 6 with the contributions of the internal CFT and of the bosonic b, c ghosts. Contrary to the AdS case discussed in [8, 10], we do not expect any non-trivial reflection coefficient in the Hpp-wave limit, so, given the well known normalization problems in the definition of two-point amplitudes, let us start considering three-point amplitudes. As it was shown in section 4.1, three-point functions in the Hpp-wave precisely agree with those resulting from the Penrose limit of three-point functions in $AdS_3 \times S^3$.

The irreducible three-point coupling can be directly extracted from the tree-point correlation functions computed in section 4. Trading the integrations over the insertion points for the volume of the $SL(2,\mathbb{C})$ global isometry group of the sphere and combining with the trilinear coupling $T_{IJK}(h_i)$ in the internal CFT one simply gets

$$\mathcal{A}^{IJK}_{abc}(\nu_i, x_i; h_i) = K_{abc}(\nu_i, x_i)C_{abc}(\nu_i)T_{IJK}(h_i) \ , \tag{7.8}$$

where $a_i = \pm, 0$, ν_i denote the relevant quantum numbers and the δ-functions associated to the conservation laws are understood. Except for $T_{IJK}(h_i)$ all the relevant pieces of information can be found in section 4. For $\mathcal{M} = \mathbf{R}^{20}$ or T^{20}, $T_{IJK}(h_i)$ is essentially purely kinematical, *i.e.* $\delta(\sum_i \vec{p}_i)$. Other consistent choices require a case by case analysis. Depending on the kinematics, amputated three-point amplitudes can be interpreted as decay or absorption rates. In particular kinematical regimes (for the charge variables) they allow one to compute mixings, to determine the $1/k \approx g_s$ corrections to the string spectrum in the Hpp-wave and to address the problem of identifying 'renormalized' BMN operators [21, 22, 64].

Additional insights can be gained from the study of four-point amplitudes. In particular the structure of their singularities provides interesting information on the spectrum and couplings of states that are kinematically allowed to flow in the intermediate channels. Needless to say, one would have been forced to discover spectral flowed states or non-highest weight states even if one had not introduced them in the external legs.

As usual, $SL(2,\mathbb{C})$ invariance allows one to fix three of the insertion points and integrate over the remaining one or rather their $SL(2,\mathbb{C})$ invariant cross ratio denoted by z in previous sections. Schematically

$$\mathcal{A}_4 = \int d^2z|z|^{\sigma_{12}-4/3}|1-z|^{\sigma_{14}-4/3}K(x_i,\nu_i)G_{Hpp}(\nu_i,x_i,z)G_{\mathcal{M}}(h_i,z) \ , \tag{7.9}$$

where, for a flat $\mathcal{M}$, $\sigma_{ij} = \kappa_{ij} + \vec{p}_i\cdot\vec{p}_j$ with κ_{ij} defined in section 4.

At present, closed form expressions for $\mathcal{A}_4$ are not available. Still the OPE allows one to extract interesting physical information. Let us consider, for a flat $\mathcal{M}$, the two cases $\mathcal{A}_{+++-}$ and $\mathcal{A}_{+-+-}$. The relevant $\hat{\mathcal{H}}_6$ four-point functions have been computed both solving the KZ equation (in section 5) and by means of the Wakimoto representation (in section 6). Expanding $\mathcal{A}_{+++-}$ in the s-channel yields

$$\begin{aligned}\mathcal{A}_{+++-} = \int d^2z|z|^{2(h_{12}-2)}\sum_{q=0}^{Q} C_{++}{}^{+}(\nu_1,\nu_2;q)C_{+-}{}^{-}(\nu_3,\nu_4;Q-q)\\ |z|^{-2q(p_1+p_2)}||x_{12}||^{2q}||x_{13}||^{2(Q-q)} + ...\end{aligned} \tag{7.10}$$

where $q=q_1$, $h_{12} = h^+(p_1+p_2,\hat{\jmath}_1+\hat{\jmath}_2)+\frac{1}{2}(\vec{p}_1+\vec{p}_2)^2$. Studying the z integration near the origin determines the presence of singularities whenever $h_{12}-q(p_1+p_2) = 1-N$ that coincides with the mass-shell condition for the intermediate state in the V^+ representation. The amplitudes $\mathcal{A}_{+-+-}$ are more interesting in that they feature the presence of logarithmic singularities in the s-channel when $p_1 = p_2$ and $p_3 = p_4$. The amplitude factorizes in the continuum of type 0 representations parameterized by s

$$\mathcal{A}_{+-+-} = \int d^2z|z|^{2(h_{12}-2)}\int s^3 ds C_{+-}{}^{0}(p_1,s)C_{+-}{}^{0}(p_3,s)|z|^{s^2}(||x_{13}||||x_{24}||)^{2k}I_{|k|} + ... \ , \tag{7.11}$$

where in the present case $h_{12} = \frac{1}{2}(\vec{p}_1+\vec{p}_2)^2$. Using the explicit form of the OPE coefficients determined in section 4, and integrating z in a small disk around the origin yields

$$\mathcal{A}_{+-+-} \approx \int d^2z|z|^{2h_{12}-4-2L}\Psi(p_1,p_3)^{L+1}|\exp\left(p_3xz + x\Psi(p_1,p_3)\right)\sum_{q=0}^{\infty}\frac{|x\Psi(p_1,p_3)|^{2q}}{q!(Q+q)!} \ , \tag{7.12}$$

where as usual $Q = L/\mu = -\sum_i \hat{\jmath}_i/\mu$ and $\Psi(p_1,p_3) = [-\log|z|^2 - 4\psi(1)-\psi(p_1)-\psi(1-p_1)-\psi(p_3)-\psi(1-p_3)]^{-1}$. For $q=Q=0$ one has

$$\mathcal{A}_{+-+-} \approx \int_{|z|<\epsilon}\frac{d|z|}{|z|^{\delta}\log|z|} \ , \tag{7.13}$$

where $\delta = 3-2h_{12}$ that converges for $\delta < 1$ but diverges logarithmically as $\mathcal{A}_{+-+-} \approx \log(h_{12}-1)$ for $\delta \approx 1$. The logarithmic branch cut departing from $h_{12} = 1$ signals the presence of a continuum mass spectrum of intermediate states with $s = 0$. Expanding in the u-channel for $p_1+p_3 = w$ one can proceed roughly in the same way and identify the continuum of intermediate states in spectral flowed type 0 representations. They signal the presence of branch cuts for each string level.

8. The holographic correspondence

Having explicit control on the detailed action of the Penrose limit on string theory in $AdS_3 \times S^3$, we can employ the original AdS_3/CFT_2 recipe to provide a concrete formula for the holographic correspondence in the Hpp-wave background. On the string side we end up with S-matrix elements as anticipated earlier [40] and defined unambiguously in [1]. On the CFT_2 side we can produce an explicit formula for the Penrose limit of CFT correlators, to be compared with the string theory S-matrix elements.

The key ingredients of such a holographic formula are:

• The original AdS_3/CFT_2 equality between "S-matrix" elements[5] for vertex operators in Minkowskian signature AdS_3 and CFT correlation functions. Introducing two charge variables $\vec{x}$ for $SL(2,\mathbb{R})$ and as many $\vec{y}$ for $SU(2)$, the "S-matrix elements" depend on both $\vec{x}$ and $\vec{y}$. On the CFT side, $\vec{x}$ represent the positions of CFT operators $\mathcal{O}_{l,\tilde{l}}(\vec{x},\vec{y})$, while $\vec{y}$ are charge variables for the $SU(2)_L \times SU(2)_R$ R-symmetry. The conformal weight of the operators $\mathcal{O}_{l,\tilde{l}}$ is given by $\Delta = l$.

• The limiting formulae (4.2.13), (4.2.14) and (4.2.16) that describe the precise way operators of the original theory map to the operators of the pp-wave theory under the Penrose contraction.

In the expressions below, $\vec{z}_i$ are the coordinates of the vertex operators on the string world-sheet, $\vec{x}_i$ are the $SL(2,\mathbb{R})$ charge variables, that represent the insertion points on the boundary, and $\vec{y}_i$ are the $SU(2)$ R-charge variables. $\Psi^{\pm}_{l}(\vec{z},\vec{x})$ are $SL(2,\mathbb{R})$ primary fields of string theory on AdS_3 corresponding to the $\mathcal{D}^{\pm}_{l}$ representations, $\Omega_{\tilde{l}}(\vec{z},\vec{x})$ are $SU(2)$ primary fields of string theory on S^3 corresponding to the $SU(2)$ representation of spin $\tilde{l}$, and $\Psi^{0}_{l,\alpha}(\vec{z},\vec{x})$ are the $SL(2,\mathbb{R})$ primary fields of string theory corresponding to the continuous representations of spin l. We neglect the internal CFT part of the operators as it is not relevant for the structure of our formulae.

The left and right charge variables $x, \bar{x}$ are related to the Cartesian ones used here by $x = x^1 + ix^2, \bar{x} = x^1 - ix^2$. Thus, the transformation that inverts the chiral

[5]These are not the standard S-matrix elements, but their closest analogue in AdS. They can be defined as the on-shell action evaluated on a solution of the (quantum) equations of motion with specified sources on the boundary. For AdS_3 such elements were conjectured by Maldacena and Ooguri [10].

charge variables, $x \to 1/x, \bar{x} \to 1/\bar{x}$ corresponds in the cartesian basis to $\vec{x} \to \vec{x}^c/|\vec{x}|^2$ where the superscript stands for a parity transformation, $(x^1, x^2)^c = (x^1, -x^2)$. Since we consider lorentzian AdS_3 also a Minkowski continuation of the charge variables is necessary, and this can readily be implemented in the CFT correlators by $x \to x^+, \bar{x} \to x^-$.

We will denote by $\mathcal{O}_{l,\tilde{l}}(\vec{x}, \vec{y})$ operators in the CFT that correspond to the appropriate ones in AdS_3

$$\Psi_l\left(\vec{z}, \vec{x}\right) \Omega_{\tilde{l}}\left(\vec{z}, \vec{y}\right) \Leftrightarrow \mathcal{O}_{l,\tilde{l}}\left(\vec{x}, \vec{y}\right) \ . \tag{8.1}$$

The AdS_3 "S-matrix elements" are functions of the spins $(l, \tilde{l})$ as well as of the charge variables $\vec{x}_i, \vec{y}_i$. They can be obtained by standard techniques by integrating the CFT correlators appropriately over the positions of the vertex operators [10]. We will split the AdS_3 states into three families, distinguished by the type of $\mathcal{H}_6$ representation they will asymptote to in the Penrose limit, namely Φ^+, Φ^- and Φ^0. Thus the starting string "S-matrix elements" are of the form

$$S^{AdS_3}_{N_{\pm,0}}(l_i, \tilde{l}_i, \vec{x}_i, \vec{y}_i | l_j, \tilde{l}_j, \vec{x}_j, \vec{y}_j | l_k, \alpha_k, \tilde{l}_k, \vec{x}_k, \vec{y}_k) \ , \tag{8.2}$$

where the index $i = 1, ..., N_+$ labels the operators that asymptote to the $\Phi^+_{p_i, \hat{\jmath}_i}$ operators, the index $j = 1, ..., N_-$ labels the operators that asymptote to the $\Phi^-_{p_j, \hat{\jmath}_j}$ operators and the index $k = 1, ..., N_0$ labels the operators that asymptote to the $\Phi^0_{s^1_k, s^2_k, \hat{\jmath}_k}$ operators. As shown in section (4), by taking the Penrose limit the $AdS_3 \times S^3$ S-matrix elements asymptote to the pp-wave S-matrix elements we computed as

$$\begin{aligned}
&\lim_{\substack{k_1 \to \infty \\ k_2 \to \infty}} \prod_{i=1}^{N_+} \left(\frac{k_1}{|\vec{x}_i|^2}\right)^{-2l_i} \left(\frac{k_2}{|\vec{y}_i|^2}\right)^{2\tilde{l}_i} \prod_{k=1}^{N_0} |\vec{x}_k|^{-2l_k} |\vec{y}_k|^{2\tilde{l}_k} \times \\
&\times S^{AdS_3}_{N_{\pm,0}} \left(l_i, \tilde{l}_i, \frac{\sqrt{k_1}\vec{x}^c_i}{|\vec{x}_i|^2}, \frac{\sqrt{k_2}\vec{y}^c_i}{|\vec{y}_i|^2} \middle| l_j, \tilde{l}_j, \frac{\vec{x}_j}{\sqrt{k_1}}, \frac{\vec{y}_j}{\sqrt{k_2}} \middle| l_k, \alpha_k, \tilde{l}_k, \vec{x}_k, \vec{y}_k \right) = \\
&= C_{N_+, N_-, N_0}(k_1, k_2) \ S^{Hpp}_{N_{\pm,0}}(p_i, \hat{\jmath}_i, \vec{x}_i, \vec{y}_i | p_j, \hat{\jmath}_j, \vec{x}_j, \vec{y}_j | s^{1,2}_k, \hat{\jmath}_k, \vec{x}_k, \vec{y}_k) \ .
\end{aligned} \tag{8.3}$$

In the previous formula the limit on the spins is taken as explained in section 4. For the first two classes of operators (labeled by i and j) we have

$$l = \frac{k_1}{2}\mu_1 p - a \ , \qquad \tilde{l} = \frac{k_2}{2}\mu_2 p - b \ , \tag{8.4}$$

with the subleading terms a and b related to $\hat{\jmath}$ in the limit as follows

$$\hat{\jmath}_i = -\mu_1 a_i + \mu_2 b_i \ , \qquad \hat{\jmath}_j = \mu_1 a_j - \mu_2 b_j \ . \tag{8.5}$$

For the third class of operators we set

$$l = \frac{1}{2} + i\sqrt{\frac{k_1}{2}} s_1 \ , \qquad \tilde{l} = \sqrt{\frac{k_2}{2}} s_2 \ , \tag{8.6}$$

and in the limit $\hat{\jmath}_k$ is given by the fractional part of the $SL(2,\mathbb{R})$ spin $\hat{\jmath}_k = -\mu_1\alpha_k$. The coefficients $C_{N_+,N_-,N_0}(k_1,k_2)$ are divergent in the limit $k_{1,2} \to \infty$ and can be computed in principle directly. Using the results obtained in section 4 we have for instance

$$C_{2,1,0}(k_1,k_2) = \sqrt{k_1 k_2} \ . \tag{8.7}$$

By employing the holographic recipe of AdS/CFT we can now write the relation between pp-wave S-matrix elements and limits of CFT correlators[6]

$$S^{Hpp}_{N_\pm}(p_i,\hat{\jmath}_i,\vec{x}_i,\vec{y}_i|p_j,\hat{\jmath}_j,\vec{x}_j,\vec{y}_j) = \lim_{\substack{k_1\to\infty \\ k_2\to\infty}} \frac{\prod_{i=1}^{N_+}\left(\frac{k_1}{|\vec{x}_i|^2}\right)^{-2l_i}\prod_{j=1}^{N_-}\left(\frac{k_2}{|\vec{y}_j|^2}\right)^{2\tilde{l}_j}}{C_{N_+,N_-}(k_1,k_2)} \times \tag{8.8}$$

$$\left\langle \prod_{i=1}^{N_+}\mathcal{O}_{l_i,\tilde{l}_i}\left(\sqrt{k_1}\frac{\vec{x}_i^c}{|\vec{x}_i|^2},\sqrt{k_2}\frac{\vec{y}_i^c}{|\vec{y}_i|^2}\right)\prod_{j=1}^{N_-}\mathcal{O}_{l_j,\tilde{l}_j}\left(\frac{\vec{x}_j}{\sqrt{k_1}},\frac{\vec{y}_j}{\sqrt{k_2}}\right)\right\rangle \ .$$

The $SL(2,\mathbb{R})$ spin is the conformal dimension of the CFT operator while the $SU(2)$ spin determines its transformation properties under the $SU(2)$ R-symmetry. The level k in the space-time CFT is interpreted as the number of $NS5$ branes used to build the background [7].

The interpretation of the limit in the CFT is as follows. CFT operators that asymptote to V^- representations (with negative values of p^+) have their position and charge variables scaled to zero. Operators that asymptote to V^+ representations (with positive values of p^+) are instead placed at antipodal points and then their positions are scaled to infinity. Finally all the spins are scaled as indicated and there is an overall renormalization. The limit of the two-point functions of the CFT is particularly simple. In this case $C_{1,1,0} = 1$ and we obtain in the Penrose limit

$$S(p_1,\hat{\jmath}_1,\vec{x}_1,\vec{y}_1|p_2,\hat{\jmath}_2,\vec{x}_2,\vec{y}_2) = \exp\left[-\mu_2 p(y_1y_2+\bar{y}_1\bar{y}_2) - \mu_1 p(x_1^+x_2^+ + x_1^-x_2^-)\right] \ , \tag{8.9}$$

where $\vec{y}_i$ are in Euclidean space and $\vec{x}_i$ are in Minkowski space .

9. Outlook

In this paper we have computed tree-level (sphere) bosonic string amplitudes in the Hpp-wave limit of $AdS_3 \times S^3 \times \mathcal{M}_{20}$ supported by NS-NS 3-form flux. For simplicity, we have only considered scalar 'tachyon' vertex operators with no excitation in the internal world-sheet CFT describing $\mathcal{M}_{20}$. The present results generalize the bosonic string amplitudes obtained for the NW model that arises in the Penrose limit of the near horizon geometry of a stack of penta-branes [1]. Preliminary results for the

[6]We ignore type V^0 operators since, although their definition and dynamics are clear on the string theory side, they are less clear in the CFT side. They are related to the continuous spectrum and the associated instabilities of the $NS5/F1$ system in analogy with the discussion in [11].

simplest 'extremal' amplitudes of the type $\langle +++- \rangle$ have been presented in [34]. We have heavily relied on current algebra techniques on the world-sheet and confirmed for the present case, with affine $\widehat{\mathcal{H}}_6$ Heisenberg symmetry, the agreement with the free-field Wakimoto realization found in [29] for the NW model, enjoying an affine $\widehat{\mathcal{H}}_4$ symmetry.

We have discussed both the $SU(2)$ symmetric case ($\mu_1 = \mu_2$) and the general case ($\mu_1 \neq \mu_2$) and observed that the corresponding exactly marginal deformations interpolate between the generic 6-d Hpp-wave ($\mu_1 \neq \mu_2$), the (super)symmetric one ($\mu_1 = \mu_2$) and the NW model ($\mu_1 = 0$ or $\mu_2 = 0$) or even flat space-time ($\mu_1 = 0$ and $\mu_2 = 0$) very much as the 'null deformation' discussed in [57] interpolates between $AdS_3 \times S^3$ and $R^+ \times S^3$ with a linear dilaton before any Penrose limit is taken. The space-time counterpart of the world-sheet RG flow is the condensation / evaporation of fundamental strings [57]. We have derived covariant bosonic string amplitudes on the sphere and shown that they are well defined even for $p^+ = 0$ states, which are difficult if not impossible to analyze in the light-cone gauge. String amplitudes expose singularities that admit a sensible physical interpretation in terms of OPE, very much as in the closely related NW model [1], and precisely match the ones resulting from the 'Saletan contraction' [26] $k_1, k_2 \to \infty$ with $\mu_1^2 k_1 = \mu_2^2 k_2$ of $SL(2,\mathbb{R})_{k_1} \times SU(2)_{k_2}$. We have thus provided further evidence for the consistency of the BMN limit [21] in this setting. A crucial role has been played by the complex charge variables that can be introduced for any group in order to compactly encode the content of (in)finite dimensional irreps [36, 37, 56, 1]. While for the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ algebra, underlying AdS_3, x and $\bar{x}$ can be viewed as coordinates on the 2-d boundary, in the case of the $\mathcal{H}_{2+2n}$ algebra, underlying a pp-wave, x^α and x_α become coordinates on a $2n$-dimensional 'holographic' screen [40] that replaces the one-dimensional null boundary representing the 'true' geometric boundary in the Penrose limit [41, 47].

At any finite but large value of k (*i.e.* the radius or any other contraction parameter) one finds (or rather expects) a perfect matching between string amplitudes in $AdS_3 \times S^3 \times \mathcal{M}_{20}$ and correlation functions in some boundary CFT$_2$. Only when the contraction is fully performed target space conformal invariance should be replaced with the relevant $\mathcal{H}_{2+2n}$ Heisenberg symmetry, along with its 'accidental' external automorphisms. For the bosonic string, there is no obvious candidate for a 'dual' boundary CFT$_2$. The naive guess would be a σ-model on a resolution of $Sym_N(\mathcal{M}_{20})$ with $N = N_1 N_{21}$ with N_1 the number of fundamental strings wrapped around an S^1 and smeared in M_{20} and N_{21} the 21-branes wrapped around $S^1 \times M_{20}$. Anyway, despite the presence of tachyons and other limitations, for states with large R-charge and corresponding to the identity sector of the world-sheet CFT describing M_{20}, tree-level (sphere) amplitudes should display the relevant pattern: conformal invariance $\to$ Saletan contraction $\to$ Heisenberg symmetry. Indeed our results at least qualitatively point in this direction.

In order to be more quantitative one should consider the (type IIB) superstring

where the candidate dual boundary CFT is the $\mathcal{N} = (4,4)$ σ-model on the (hyperkähler resolution of) $Sym_N(\mathcal{M}_4)$ with $N = N_1 N_5$ with N_1 the number of fundamental strings wrapped around an S^1 and smeared in M_4 and N_5 the number of NS5 branes wrapped around $S^1 \times M_4$ [17]. Despite some initial success for the matching of the KK supergravity spectrum with the spectrum of chiral primary operators [58], a stringy exclusion principle [48], which is related to the existence of a maximal allowed R-symmetry charge, even for multi-particle states (differently from the more familiar CFT$_4$ case!) has stimulated some reconsideration. In particular, it is widely believed that the 'symmetric orbifold point' of the boundary CFT$_2$, that should be the analogue of the 'higher spin enhancement point' $g_{YM} = 0$ of $\mathcal{N} = 4$ SYM in $D = 4$ [68, 69, 70], does not coincide with the locus in the moduli space where the string description is under control, at least in the case of NS-NS flux [5, 6]. The latter should in fact correspond to a singular CFT$_2$ due to the presence of non-compact directions in the target space of the σ-model related to the possibility of string emission from penta-branes [11], as mentioned above. Indeed the presence of a continuous spectrum of long strings in AdS_3 and their images under spectral flow seems to point in this direction [8, 9, 10]. In particular some of the missing chiral primaries [13, 14], usually associated to short strings, may have reached the unitary bound, re-combined with other states with the proper quantum numbers and disappeared in the continuum as long strings. Indeed, following the analysis of [39, 38], *i.e.* extrapolating the string spectrum to the relevant 'enhancement / orbifold' point, it has been recently argued that this generalized Higgs mechanism is at work [33]. Considerations of the dynamics in pp-wave backgrounds [45, 44, 51, 63] have certainly helped pursuing this line of thought. Once again, long strings are associated with states with $p^+ \in \mathbb{Z}$ which require a covariant description, being related by spectral flow to the $p^+ = 0$ representations.

Alternatively, one may consider turning on R-R fluxes which should effectively compactify the target space [11, 8, 10]. The hybrid formalism of Berkovits, Vafa and Witten [15] seems particularly suited to this purpose as it allows the computation of string amplitudes, at least for the massless modes [16], and the study of the Penrose limit in a covariant way [52]. The pure spinor formalism [4] might be needed for $\mathcal{M}_4 = T^4$ due to the enhanced susy ($16 \to 24$). The mismatch for 3-point functions of chiral primaries (or rather their superpartners)[7] [72, 71] calls for additional investigation in this direction and a careful comparison with the boundary CFT results of [51]. Once again, the BMN limit [21] may shed some light on this issue as well as on the short-distance logarithmic behavior found in [10] for AdS and in [1] for NW that require a resolution of the operator mixing along the lines of [64] or a scattering matrix interpretation [62].

[7] This may actually be due to the reduced number of susy's (16 instead of 32 !) and the consequent lack of a non-renormalization theorem for these couplings.

It would thus be very interesting and important to extend the present analysis to the superstring, compute scattering amplitudes in the Hpp-wave and study their (super)symmetry properties. In principle, one would like to address some of the above issues (spectrum, trilinear couplings and operator mixing) in a more quantitative way and possibly reformulate the holographic duality in the Penrose limit directly in terms of propagators along the lines of [51, 46] that should further clarify the role of the charge variables as coordinates in a holographic screen [35].

In summary, we would like to argue that the BMN limit of physically sensible correlation functions is well defined and perfectly consistent, at least for the CFT dual to $AdS_3 \times S^3$. In particular it should not lead to any of the difficulties encountered in the case of $\mathcal{N} = 4$ SYM as a result of the use of perturbative schemes or of the light-cone gauge. The case of the Hpp-wave supported by NS-NS fluxes being under control at each step (before and after the Penrose limit is taken) could prove to be a source of extremely useful insights in holography and the duality between string theories and field theories.

Acknowledgments

The authors are grateful to O. Aharony, I. Bakas, N. Berkovits, N. Constable, A. Giveon, C. Hull, J. F. Morales, K. Panigrahi, I. Plefka, E. Sezgin, K. Sfetsos, E. Sokatchev, J. Sonnenschein, Ya. Stanev for discussions.

This work was partially supported by EU contracts HPRN-CT-2000-00122, HPRN-CT-2000-00148, HPRN-CT-2000-00131, MIUR-COFIN contracts 2001-025492, 2003-023852, INFN, INTAS contract N 2000-254, NSF grant PHY99-07949, and the NATO contract PST.CLG.978785. The work of G. D. is supported by a European Commission Marie Curie Postdoctoral Fellowship, contract HMPF CT 2002-01908. The work of O. Z. was also supported by a ICSC-World Laboratory fellowship.

E. K. would like to thank KITP for hospitality during part of this work. M. B. and O. Z. would like to thank LPTHE at Jussieu for hospitality during completion of this work.

APPENDIX

A. The σ-model view point

Elements of the $\mathbf{H}_6$ Heisenberg group can be parametrized as [29]

$$g(u, v, \gamma^\alpha, \bar{\gamma}_\alpha) = e^{\frac{\gamma^\alpha}{\sqrt{2}} P_\alpha^+} e^{uJ - vK} e^{\frac{\bar{\gamma}_\alpha}{\sqrt{2}} P^{-\alpha}} \ . \tag{A.1}$$

As usual the σ-model action can be written in terms of the Maurer-Cartan forms and reads

$$S = \frac{1}{2\pi} \int d^2\sigma \left(-\partial u \bar{\partial} v + \sum_{\alpha=1}^{2} e^{i\mu_\alpha u} \partial \bar{\gamma}_\alpha \bar{\partial} \gamma^\alpha \right) , \tag{A.2}$$

where we have used $\langle J, K \rangle = 1$ and $\langle P_\alpha^+, P^{-\alpha} \rangle = 2$. The metric and B field are then given by

$$ds^2 = -2dudv + 2 \sum_\alpha e^{i\mu_\alpha u} d\gamma^\alpha d\bar{\gamma}_\alpha \ , \tag{A.3}$$

$$B = -du \wedge dv + \sum_\alpha e^{i\mu_\alpha u} d\gamma^\alpha \wedge d\bar{\gamma}_\alpha \ . \tag{A.4}$$

Two auxiliary fields β_α and $\bar{\beta}^\alpha$, defined by the OPE's

$$\beta_\alpha(z) \gamma^\beta(w) \sim \frac{\delta_\alpha^\beta}{z - w} \ . \tag{A.5}$$

complete the ghost-like systems that appear in the Wakimoto representation.

With the help of β_α and $\bar{\beta}^\alpha$, the action can be written as

$$S = \frac{1}{2\pi} \int d^2z \left(-\partial u \bar{\partial} v + \sum_{\alpha=1}^{2} [\bar{\beta}^\alpha \partial \bar{\gamma}_\alpha + \beta_\alpha \bar{\partial} \gamma^\alpha - e^{-i\mu_\alpha u} \beta_\alpha \bar{\beta}^\alpha] \right) \ , \tag{A.6}$$

that gives us back (A.2) upon using the equations of motion for β_α and $\bar{\beta}^\alpha$.

In the Wakimoto representation, the currents can be written as [29]

$$\begin{aligned} P_\alpha^+(z) &= -\beta_\alpha(z), \\ P^{-\alpha}(z) &= -2(\partial \gamma^\alpha + i \partial u \gamma^\alpha)(z), \\ J(z) &= -(\partial v - i \textstyle\sum_\alpha \mu_a \beta_\alpha \gamma^\alpha)(z), \\ K(z) &= -\partial u(z) \ , \end{aligned} \tag{A.7}$$

in agreement with the result of section 6. A simple identification of the $\mathbf{H_6}$ group parameters and the string coordinates, recast the metric in the more standard form of (2.5). Generalizing the results of [29], it is easy to show that string coordinates and Wakimoto fields are related as follows

$$\begin{aligned} u(z, \bar{z}) &= u(z) + \bar{u}(\bar{z}), \\ v(z, \bar{z}) &= v(z) + \bar{v}(\bar{z}) + 2i \bar{\gamma}_{L\alpha}(z) \gamma_R^\alpha(\bar{z}), \\ w^\alpha(z, \bar{z}) &= e^{-i\mu_\alpha u(z)} [e^{i\mu_\alpha u(z)} \gamma_L^\alpha(z) + \gamma_R^\alpha(\bar{z})], \\ \bar{w}_\alpha(z, \bar{z}) &= e^{+i\mu_\alpha u(z)} [\bar{\gamma}_{L\alpha}(z) + e^{i\mu_\alpha \bar{u}(\bar{z})} \bar{\gamma}_{R\alpha}(\bar{z})] \ . \end{aligned} \tag{A.8}$$

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